THE INFIMUM OF THE VOLUMES OF CONVEX POLYTOPES OF ANY GIVEN FACET AREAS IS 0

N. V. ABROSIMOV*, E. MAKAI, JR.**, A. D. MEDNYKH*,
YU. G. NIKONOROV*, G. ROTE

ABSTRACT. We prove the theorem mentioned in the title, for $\mathbb{R}^n$, where $n \geq 3$. The case of the simplex was known previously. Also, the case $n = 2$ was settled, but there the infimum was some well-defined function of the side lengths. We also consider the cases of spherical and hyperbolic n-spaces. There we give some necessary conditions for the existence of a convex polytope with given facet areas, and some partial results about sufficient conditions for the existence of (convex) tetrahedra.

2010 Mathematical Subject Classification: 52B11 (primary), 52A38, 52A55 (secondary).

Keywords and phrases: convex polytopes, volume, Euclidean, spherical, hyperbolic spaces.

1. PRELIMINARIES

Minimum-area convex polygons with given side lengths are characterized by the following theorem.

Theorem A ([11])**.** *Let $m \geq 3$, and $s_m \geq s_{m-1} \geq \ldots \geq s_1 > 0$, and $s_m < s_{m-1} + \ldots + s_1$. Then the infimum of the areas of convex m-gons in $\mathbb{R}^2$, having side lengths s_i, is the minimal area of those triangles, whose sides have lengths $\sum_{i \in I_1} s_i$, $\sum_{i \in I_2} s_i$, $\sum_{i \in I_3} s_i$ (supposing these lengths satisfy the non-strict triangle inequality), where $\{I_1, I_2, I_3\}$ is an arbitrary partition of $\{1, \ldots, m\}$ into non-empty parts. If the cyclic order of the sides is fixed, then an analogous statement holds, where the sides with indices in I_j (for each $j \in \{1, 2, 3\}$) form an arc of the polygonal curve.*

When the convexity assumption is dropped, we have the following result.

Theorem B ([11, 36])**.** *Let $m \geq 3$, let $s_m \geq s_{m-1} \geq \ldots \geq s_1 > 0$, and let $s_m < s_{m-1} + \ldots + s_1$. Then the infimum of the areas of simple m-gons in $\mathbb{R}^2$, having side lengths s_i, is the minimal area of those triangles, whose sides have lengths $\sum_{i \in I_1} \varepsilon_i s_i$, $\sum_{i \in I_2} \varepsilon_i s_i$, $\sum_{i \in I_3} \varepsilon_i s_i$ (supposing these lengths are non-negative, and satisfy the non-strict triangle inequality), where the ε_i's are arbitrary signs, and $\{I_1, I_2, I_3\}$ is an arbitrary partition of $\{1, \ldots, m\}$ into non-empty parts. (Moreover, if this minimum is not 0, we may additionally suppose that, for each $j \in \{1, 2, 3\}$, the sum $\sum_{i \in I_j} \varepsilon_i s_i$*

1

cannot be written as $\sum_{i \in I'_j} \varepsilon_i s_i + \sum_{i \in I''_j} \varepsilon_i s_i$, where $\{I'_j, I''_j\}$ is a partition of I_j, and where both these partial summands are positive.)

We remark that the proofs in the two papers were different. Also, in [36] the result is formulated in a special case only, but all the ingredients of the proof of the general case are present in [36] as well.

In our paper we write $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$, for the *Euclidean, hyperbolic, or spherical n-space*, respectively. Theorems A and B extend to $\mathbb{S}^2$ and $\mathbb{H}^2$ as follows:

Theorem C ([11]). *Let $m \geq 3$, let $s_m \geq s_{m-1} \geq \ldots \geq s_1 > 0$, and let $s_m < s_{m-1} + \ldots + s_1$. Rather than $\mathbb{R}^2$, we consider $\mathbb{H}^2$ and $\mathbb{S}^2$, but, in case of $\mathbb{S}^2$, we additionally suppose $\sum_{i=1}^{m} s_i \leq \pi$. Then, for both cases, the word-for-word analogues of Theorems 1 and 2 hold for $\mathbb{H}^2$ and $\mathbb{S}^2$.*

In each of these three theorems, the question of finding the infimum is reduced to finding the minimum of a set of non-negative numbers, the cardinality of this set being bounded by a function of m. (For Theorem A, or B, this bound is 3^m, or 6^m, respectively. For Theorem A, for given cyclic order of the sides, this bound is $\binom{m}{3}$. For Theorem C, the respective bounds are valid.)

Böröczky et al. [11] posed the question whether it is possible to extend these theorems to dimensions $n \geq 3$. Their conjecture was that, analogously to the two-dimensional case, the solutions would be given as the volumes of some simplices. Unfortunately, they were unaware of the fact that the case of simplices already had long ago been solved, namely in 1938, cf. references later in this paper.

The analogous problem, about the maximal volume of simplices with given facet areas (an isoperimetric type problem), was solved even much longer ago: in 1773 by Lagrange [31] for $\mathbb{R}^3$, and in 1866 by Borchardt [9] for $\mathbb{R}^n$. A simplex is called *orthocentric*, if it has an orthocentre, i.e., a common point of all altitudes. It can be characterized also as a simplex with any two disjoint edges (or, equivalently, disjoint positive dimensional faces) being orthogonal. For this reason an orthocentric simplex is sometimes also called *orthogonal*, although orthocentric is the presently used terminology. For a relatively recent exposition of the above mentioned facts, and some other properties of orthocentric simplices, cf. e.g. [21]. Cf. also the very recent paper [17], whose first part is a comprehensive survey about orthocentric simplices. In [17] it is also stressed that, for many elementary geometrical theorems, the objects in $\mathbb{R}^n$, corresponding to triangles, are not the general simplices, but just the orthocentric ones.

Theorem D ([31, 9]). *Let $n \geq 3$. Then among the simplices in $\mathbb{R}^n$, with given facet areas $S_{n+1} \geq \ldots \geq S_1 > 0$ (if such simplices exist), there is (up to congruence) exactly one simplex of maximal volume. It is also the (up to congruence) unique orthogonal simplex with these facet areas.*

Unaware of the above mentioned solution of the maximum problem, in 1937, A. Narasinga Rao posed the following problem [42]:

> "The areas of the four facets of a tetrahedron are $\alpha, \beta, \gamma, \delta$. Is the volume determinate? If not, between what limits does it lie?"

This problem was soon solved independently in the papers [53, 26, 6, 27]. In fact, under the above hypothesis, the volume is not determined (if such tetrahedra

exist, which we suppose). Moreover, they reproved that there is, up to congruence, exactly one tetrahedron of maximal volume with the given face areas, which is also orthogonal — and they reproved that it is also the unique orthocentric simplex with these facet areas — and there is a tetrahedron with volume as small as we want. A generalization of their first mentioned result to multi-dimensional Euclidean spaces was obtained in [53, 26, 27]. We cite only their result about the infimum of the volumes.

Theorem E ([53, 26, 6, 27]). *Let $n \geq 3$, and let $S_{n+1} \geq S_n \geq \ldots \geq S_1 > 0$. Then there is a simplex in $\mathbb{R}^n$ with these $(n-1)$-volumes of the facets, if and only if*

$$S_{n+1} < S_1 + S_2 + \ldots + S_n.$$

Suppose that this inequality holds. Then, among the simplices in $\mathbb{R}^n$, with facet areas $S_1, S_2, \ldots, S_{n+1}$, we can find, for any $\varepsilon > 0$, a nondegenerate simplex with volume at most ε.

The proof of Theorem E in [27] was based on the following statement, valid just for simplices only. Let T be a simplex with facet areas $S_1, S_2, \ldots, S_{n+1}$, and respective facet outer unit normals $u_1, \ldots, u_{n+1}$; then $\sum_{i=1}^{n+1} S_i u_i = 0$. Let us consider an $(n+1)$-gon $P \subset \mathbb{R}^n$ with side vectors $S_1 u_1, \ldots, S_{n+1} u_{n+1}$. Its convex hull T' is then also a simplex, whose volume is invariant under permutations of the side vectors of P. Moreover, for the volumes of T and T' we have $V(T)^{n-1} = V(T')[(n-1)!]^2/n^{n-2}$. [27, p. 306]. Knowing this, [27] could make $V(T')$ arbitrarily small, based on some calculations. However, this can also be done by choosing $u_1, \ldots, u_{n+1}$ in a small neighbourhood of the $x_1 \ldots x_{n-1}$-coordinate hyperplane. Cf. also the first and third proofs of our Theorem 2.

The question of maximal volume for polytopes, with given facet areas, is much less understood.

As for positive results, by [10, Theorem 2], among non-degenerate polytopes in $\mathbb{R}^n$, with given facet areas and facet outer unit normals, the maximal volume is attained for the (up to translation) unique convex polytope with these given facet areas and facet outer unit normals. (For possibly coinciding facet outer unit normals one has to add their areas. Also, [10, Theorem 3] asserts an analogue of this theorem, for compact sets $C \subset \mathbb{R}^n$ that are equal to the closures of their interiors, with ∂C being a C^1-submanifold.)

For any given number m of facets the maximum volume, for given total surface area, is attained when the polytope has an inball, moreover the facets touch the inball at their centroids (Lindelöf's theorem, [48, p. 43], cf. also [19, II.4.3, p. 264]).

Now let us restrict our attention to $\mathbb{R}^3$. [18, Theorem 1, p. 175] (cf. also [19, II.4.3, p. 265, Satz]) asserts that among (convex) polyhedra with given surface area, and $m = 4, 6$ or 12 faces, the maximal volume is attained for the regular tetrahedron, cube, or regular dodecahedron, respectively. His inequality (valid for each m), is also asymptotically sharp, for $m \to \infty$. For $m = 5$, the extremal polyhedron is the regular triangular prism, having an inball, cf. [48, p. 41]. However, for $m = 8, 20$, the extremal polyhedron is not the regular octahedron, or icosahedron, respectively, [22, p. 234]. For an old, or a recent survey about this isoperimetric problem, for convex polyhedra in $\mathbb{R}^3$, with given number of faces, cf. [22], or the introduction of [49], respectively.

Also, [3, Ch. 3, §3, **2**, p. 150, Satz 1] says that for convex polyhedra $P, Q \subset \mathbb{R}^3$, any mapping of ∂P to ∂Q, preserving the geodesic distance of any pair of points of ∂P (i.e., the length of the shortest arc in ∂P, joining these points) extends to an isometry of $\mathbb{R}^3$. (For convex bodies $P, Q \subset \mathbb{R}^3$, where ∂P is of class C^2, the analogous theorem holds, cf. [2, Ch. 8, §5, p. 337] for a special case, and [39, Introduction, §1, A, p. 8, Theorem 1, and Ch. 3, **3**, p. 66, Theorem 1], for the above described general case.)

However, this unicity theorem does not say that this unique convex polyhedron would have the maximal volume. Just conversely: taking another model, let us imagine that the facets of a polyhedron are rigid, the edges of the polyhedron behave like hinges, so that the dihedral angles can be varied freely. Simple physical models show that if, e.g., we take a cube, and subdivide its facets suitably to smaller convex polygonal faces, preserving the symmetries of the cube, then, by blowing up the model, the volume increases. The symmetries of the cube are preserved, the facial diagonals become tight, and are a bit curved outwards, which implies that the vertices come closer to the centre of the cube. Therefore the edges shorten, which implies that they become crumpled, with wrinkles perpendicular to the original edge. Globally, the polyhedron becomes more "ball-like". This volume increasing phenomenon was observed first by A. V. Pogorelov in the theory of thin shells in mechanics, for shells of form a convex body, cf. [40], [41]. These facts can be proved also mathematically, namely that there exist non-convex polyhedra, with the same symmetry group, having the prescribed system of faces, with hinges at the edges, and having a larger volume than the cube. The relative volume increase for the cube can be $1.2403\ldots$. Cf. the papers [33], [13], [8], [4], [37] (who showed that each convex polyhedron in $\mathbb{R}^3$ can be deformed, preserving the geodesic distances, and increasing the volume), [34], [50] (where "inextensional" is the engineer's word for "isometric, w.r.t. the geodesic distance"), [51], which papers deal also with other regular, and non-regular convex polyhedra. (In [34], according to a private communication from the second named author of [34], in the tableau summarizing the numerical results, pp. 154, 181, the values in the middle coloumn, for the dodecahedron and the icosahedron, are not correct — actually they are less than the values in the third column, which are proved in [34], and which are the best published values.) The relative volume increase for the cube can be $1.2403\ldots$, cf. [34]. This value is improved in [51] to $1.2461\ldots$. For a recent survey on this and related questions cf. [43].

Notations. Here, and everywhere in the paper, $V(\cdot)$ denotes volume of a set, $S(\cdot)$ its surface area, $\operatorname{diam}(\cdot)$ its diameter, $\operatorname{aff}(\cdot)$ its affine hull, $\operatorname{lin}(\cdot)$ its linear hull, and $\partial(\cdot)$ its boundary. If we want to stress also the dimension n, we sometimes will write $V_n(\cdot)$ for the n-volume. Sometimes we will refer to the $(n-1)$-volume in $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$ as *area*. We write κ_n for the volume of the unit ball in $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$, we write $[x, y]$, or, for x, y distinct, $\ell(x, y)$, for the the segment, or line, joining x and y (except for antipodes on $\mathbb{S}^n$, and on $\mathbb{S}^n$ we mean by $[x, y]$ the shorter such segment), and $|xy|$ denotes the distance of x and y.

2. New results

2.1. **Euclidean Space.** The following Theorem 1 can be considered as folklore, whose proof we could not locate. Therefore, for completeness, we state and prove it.

Theorem 1. *Suppose that $m > n \geq 3$ are integers, and consider any sequence of numbers $S_m \geq S_{m-1} \geq \ldots \geq S_1 > 0$. Then the following conditions are equivalent:*

   (i) *There is a non-degenerate polytope $P \subset \mathbb{R}^n$ with m facets, and with facet areas $S_1, S_2, \ldots, S_m$;*
   (ii) *There is a non-degenerate convex polytope $P \subset \mathbb{R}^n$ with m facets, and with facet areas $S_1, S_2, \ldots, S_m$;*
   (iii) *The inequality $S_m < S_1 + S_2 + \ldots + S_{m-1}$ holds.*

If, in (i) *or* (ii)*, we allow also degenerate polytopes, then they imply, rather than* (iii)*,*

   (iii′) *$S_m \leq S_1 + \ldots + S_{m-1}$, with equality if and only if the polytope degenerates into the doubly counted facet with area S_m.*

Theorem 2. *Let $m > n \geq 3$ be integers. For any $\varepsilon > 0$ and for every sequence of numbers $S_m \geq S_{m-1} \geq \ldots \geq S_1 > 0$ such that $S_m < S_1 + S_2 + \ldots + S_{m-1}$, there is a non-degenerate convex polytope $P \subset \mathbb{R}^n$ with m facets, with facet areas $S_1, S_2, \ldots, S_m$, and with volume $V(P) \leq \varepsilon$.*

We give three different proofs of Theorem 2. The first one is independent of Theorem D, and reproves the case of the simplex; it is an existence proof, by contradiction. The second proof uses Theorem D; we reduce the question to the case of simplices. Both proofs rely on delicate convergence arguments (see Sections 3 and 4). The third proof is geometric. It constructs examples with small volumes that are like "needles". In particular, we will give an explicit upper bound for the volumes of our examples, in terms of the "steepness" of its facets (Lemmas 2 and 4). If we consider n and the facet areas as fixed, our estimate is sharp up to a constant factor (see Lemma 4).

Thus, there is a very interesting dichotomy. In Theorems A and B, for $\mathbb{R}^2$ (and also in Theorem C for $\mathbb{H}^2$ and $\mathbb{S}^2$), we have some definite functions of the side lengths as infima. In Theorem 2, for $\mathbb{R}^n$, with $n \geq 3$, the infimum does not depend at all on the facet areas.

2.2. **Hyperbolic Space.** For the hyperbolic case, we have a word for word analogue of the implications (ii)$\Longrightarrow$(iii) and (ii)$\Longrightarrow$(iii') from Theorem 1 (under the respective hypotheses).

Proposition 1. *Let $P \subset \mathbb{H}^n$ be a polytope, with facet areas $S_m \geq S_{m-1} \geq \ldots \geq S_1 > 0$. Then the inequality $S_m \leq \sum\limits_{i=1}^{m-1} S_i$ holds, with equality if and only if P degenerates into the doubly counted facet with area S_m.*

Next we give two statements, showing that the necessary condition in Proposition 1, together with the inequalities $S_i \leq \pi$, is not sufficient for the existence even of a tetrahedron in $\mathbb{H}^3$, with these facet areas. In other words, there are some further necessary conditions.

Proposition 2. *If we admit polyhedra in $\mathbb{H}^3$, with all vertices distinct, but possibly having some infinite vertices, then a polyhedron with facet areas $S_m, S_{m-1}, \ldots, S_3$ maximal (i.e., $(k-2)\pi$ for a k-gonal face), while S_2, S_1 not maximal, does not exist.*

Example 1. This would suggest that, for polyhedra in $\mathbb{H}^3$, if all facets but two have areas nearly maximal (i.e., nearly $(k-2)\pi$ for a k-gonal face), then the same would hold for the remaining two facets as well. However, this is not true; even in the convex case, these two facets can have areas close to 0. Consider a very large circle in $\mathbb{H}^2 \subset \mathbb{H}^3$, and a regular l-gon $p_1 \ldots p_l$ inscribed to it ($l \geq 3$). Let us choose p_{l+1} on our circle, with $|p_l p_{l+1}| = \varepsilon$. Then all triangles with vertices among the p_i's have areas close to π, except those that contain both p_l and p_{l+1}, which have very small areas. Now perturb these points p_i a little bit in $\mathbb{H}^3$, so that no four of them lie in a plane. Then their convex hull is a triangle-faced convex polyhedron, and the perturbation of the segment $[p_l, p_{l+1}]$ is an edge of it. (Use the collinear model. For any convex polygon with strictly convex angles, after this perturbation, its edges will remain edges of the convex hull.) The two facets of our polyhedron incident to this edge have very small areas, while all other facets have areas close to π, i.e., are nearly maximal.

However, an analogous statement for all but one facets will be shown, for the convex case.

Proposition 3. *Suppose that we have a convex polyhedron in $\mathbb{H}^3$, with infinite vertices admitted, having facet areas $S_m \geq \ldots \geq S_2 \geq S_1 > 0$, these facets being a k_m-gon, $\ldots, k_1$-gon, respectively. Then, for any $i \in \{1, \ldots, m\}$, we have*

$$(k_i - 2)\pi - S_i \leq \sum_{\substack{1 \leq j \leq m \\ j \neq i}} ((k_j - 2)\pi - S_j).$$

If there is a finite vertex, with incident edges not in a plane, then this inequality is strict.

In §6, Remark 9, it will be explained that in a sense there are no interesting analogues of Proposition 3 for $\mathbb{R}^3$ and $\mathbb{S}^3$.

Now we turn to sufficient conditions for the existence of hyperbolic tetrahedra.

Theorem 3. *Let the numbers $S_4 \geq S_3 \geq S_2 \geq S_1 > 0$ be such that $S_4 < \pi/2$, and $S_4 < S_1 + S_2 + S_3$, and one of the inequalities*

$$\tan(S_1/2) > \frac{1 - \cos S_4}{2\sqrt{\cos S_4}}, \tag{1}$$

and

$$S_4 \geq S_3 + S_2 \tag{2}$$

holds. Then there is a non-degenerate tetrahedron $T \subset \mathbb{H}^3$ with facet areas S_1, S_2, S_3, S_4.

Remark 1. Since $S_4 \geq S_1$, the following inequality holds under assumption (1) of Theorem 3:

$$\sqrt{\frac{1 - \cos S_4}{1 + \cos S_4}} = \tan(S_4/2) \geq \tan(S_1/2) > \frac{1 - \cos S_4}{2\sqrt{\cos S_4}}$$

From this we get $S_4 < b := \arccos(\sqrt{5} - 2) = 1.3324788\ldots$. Note also that, for $\arccos(\sqrt{5} - 2) > S_4 > a := 0.6228644\ldots$, from the inequality $\tan(S_1/2) > (1 - \cos S_4)/(2\sqrt{\cos S_4})$ we get $S_1/2 > S_4/6$ — that amounts to the inequality

$$\frac{1 - \cos S_4}{2\sqrt{\cos S_4}} > \tan \frac{S_4}{6}, \tag{3}$$

for $a < S_4 \leq b$ — that implies $S_4 < S_1 + S_2 + S_3$. (To show (3), we square it, and writing $x := S_4$, we obtain

$$f(x) := \frac{(1 - \cos x)^2}{4 \cos x} > \tan^2 \frac{x}{6} =: g(x),$$

for $a < x \leq b$. We note that a was chosen so as to satisfy $f(a) = g(a)$. Then, for $a < x \leq 1$, we have

$$f'(x) = (\sin x \cdot \tan^2 x)/4 \geq (\sin a \cdot \tan^2 a)/4 = 0.0752352\ldots,$$

and

$$g'(x) = \frac{\tan(x/6)}{3 \cos^2(x/6)} \leq \frac{\tan(1/6)}{3 \cos^2(1/6)} = 0.0576627\ldots.$$

Similarly, for $1 \leq x \leq b$, we have

$$f'(x) \geq (\sin 1 \cdot \tan^2 1)/4 = 0.5102509\ldots,$$

and

$$g'(x) \leq \frac{\tan(b/6)}{3 \cos^2(b/6)} = 0.0791058\ldots.$$

Hence, for $a < x \leq b$, we have $f'(x) - g'(x) > 0$, therefore also $f(x) - g(x) > 0$, as asserted.)

2.3. **Spherical Space.** For the spherical case, we give some necessary and some sufficient conditions for the existence.

We say that a set $X \subset \mathbb{S}^n$ (for $n \geq 2$) is *convex*, if for any two non-antipodal $x, y \in X$, the connecting shorter great-$\mathbb{S}^1$-arc also belongs to X. Since our investigated sets will contain non-trivial arcs, the case of an antipodal pair of points in X can be excluded: this property will be equivalent to saying that for any $x, y \in X$, at least one connecting not-longer great-$\mathbb{S}^1$-arc also belongs to X. By a *simplex in* $\mathbb{S}^n$ we mean the set of those points of $\mathbb{S}^n$, which have non-negative coordinates in some (non-orthogonal) coordinate system, with origin at 0, with its usual face lattice system, and limiting positions of these. Thus, e.g., we will not consider concave spherical triangles, or spherical triangles with sides $3\pi/2, \pi/4, \pi/4$, or $2\pi, 0, 0$. For $\mathbb{R}^n$, and by the collinear model, for $\mathbb{H}^n$, all simplices are convex. Observe that an open half-$\mathbb{S}^n$ also has a collinear model in $\mathbb{R}^n$, also respecting convexity: for the open southern half-$\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ consider the central projection to the tangent $\mathbb{R}^n$ at the South Pole. So a convex combinatorial simplex in an open half-$\mathbb{S}^n$ is among the simplices that we considered above.

Proposition 4. *Let $P \subset \mathbb{S}^n$ be a polytope, with facet areas $S_m \geq \ldots \geq S_1 > 0$, all of whose facets lie in some closed half-$\mathbb{S}^{n-1}$'s. Then*

$$S_m \leq S_1 + \ldots + S_{m-1},$$

where strict inequality holds if P is contained in an open half-$\mathbb{S}^n$, and does not degenerate into the doubly counted facet with area S_m.

If P is a convex polytope contained in some closed half-$\mathbb{S}^n$, then

$$\sum_{i=1}^{m} S_i \leq V_{n-1}(\mathbb{S}^{n-1}),$$

where strict inequality holds if P is contained in an open half-$\mathbb{S}^n$.

Remark 2. Clearly, in the first inequality of Proposition 4, the hypothesis that each facet lies in some closed half-$\mathbb{S}^n$ cannot be dispensed. Even we may have a degenerate combinatorial simplex, lying in some great-$\mathbb{S}^{n-1}$, with one facet strictly containing a half-$\mathbb{S}^{n-1}$, already for $n = 2$. Also, in the second inequality of Proposition 4, if P is contained in a closed half-$\mathbb{S}^n$, but is not contained in an open half-$\mathbb{S}^n$, the case of equality can occur, if P degenerates, one facet is a closed half-$\mathbb{S}^{n-1}$, and the other facets have union this closed half-$\mathbb{S}^{n-1}$, or the closure of its complement in this $\mathbb{S}^{n-1}$.

Remark 3. In $\mathbb{H}^3$ and $\mathbb{S}^3$ the question, for which S_i is there a tetrahedron (for $\mathbb{S}^3$ meant as above described) with facet areas S_i, is algorithmically decidable. In other words, one can describe the set of those quadruples (S_1, S_2, S_3, S_4), which are the facet areas of some tetrahedron in $\mathbb{H}^3$ or $\mathbb{S}^3$, by using a finite set of equations and inequalities for some rational functions of the $\tan(S_i/2)$'s with rational coefficients, together with the usual logical connectives "and", "or", "not". (Such a set is a *semi-algebraic set* in the variables $\tan(S_i/2)$.)

In fact, we may use polar coordinates, and use trigonometric functions of the angles, and hyperbolic/trigonometric functions of the sides (in $\mathbb{H}^3$ or $\mathbb{S}^3$, respectively). Then we write, that there exist four vertices, given with their respective coordinates, such that the facet areas of the corresponding tetrahedron should be S_i, for $1 \leq i \leq 4$. The resulting formula is a so-called *elementary formula* in the language of $\mathbb{R}$, cf. [52]. By [52], this formula is equivalent to a one described above under the name semi-algebraic set, in the variables $\tan(S_i/2)$.

However, because of the complexity of the decision algorithm, possibly the formula obtainable this way would be very long.

Remark 4. The same cannot be said about $\mathbb{H}^n$ and $\mathbb{S}^n$, for $n \geq 4$. Here, the $(n-1)$-volumes of the facets are transcendental functions. For $\mathbb{S}^n$ we have compactness if we also admit simplices in a closed half-$\mathbb{S}^n$, but not in an open half-$\mathbb{S}^n$. The case $S_1 = 0$ can be described: then there is a partition of the other facets into two classes, such that for the two classes the sums of the facet areas are equal. For $\mathbb{H}^n$, one may expect something similar; however, there, even taking into consideration simplices with possibly infinite vertices, we cannot use compactness considerations, since the volume is not continuous when at least two vertices are allowed to be infinite.

Theorem 4. *Let the numbers $S_4 \geq S_3 \geq S_2 \geq S_1 > 0$ be such that $S_4 \leq \pi/2$, and $S_4 < S_1 + S_2 + S_3$, and one of the inequalities*

$$\tan(S_1/2) \geq \frac{1 - \cos S_4}{2\sqrt{\cos S_4}}, \tag{4}$$

and

$$S_4 \geq S_3 + S_2 \tag{5}$$

holds. Then there is a non-degenerate (convex) tetrahedron $T \subset \mathbb{S}^3$, with facet areas S_1, S_2, S_3, S_4.

Remark 5. Analogously to Remark 1, with the notations used there, we get $S_4 \leq b$, and, for $a < S_4 \leq b$, we get $S_4 < S_1 + S_2 + S_3$.

Now we turn to sufficient conditions for the existence of spherical polyhedral complexes. The second statement of Proposition 5 says that for combinatorial simplices, contained in some closed half-$\mathbb{S}^n$, the two necessary conditions from Proposition 4 for their existence are also sufficient for their existence.

Proposition 5. *Let $n \geq 2$ and $m \geq 3$ be integers, and let $S_m \geq \ldots \geq S_1 > 0$, with $S_m \leq S_1 + \ldots + S_{m-1}$, and $S_1 + \ldots + S_m \leq V_{n-1}(\mathbb{S}^{n-1})$. Then there exists a convex n-dimensional polyhedral complex in $\mathbb{S}^n$, lying in a closed half-$\mathbb{S}^n$, with facet areas $S_1, \ldots, S_m$. All of its facets have two $(n-2)$-faces. For $S_m < S_1 + \ldots + S_{m-1}$ and $S_1 + \ldots + S_m < V_{n-1}(\mathbb{S}^{n-1})$, we have that all its dihedral angles are less than π.*

Let $n \geq 2$ be an integer, and let $S_{n+1} \geq \ldots \geq S_1 > 0$, with $S_{n+1} \leq S_1 + \ldots + S_n$, and $S_1 + \ldots + S_{n+1} \leq V_{n-1}(\mathbb{S}^{n-1})$. Then there exists a convex polyhedral complex in $\mathbb{S}^n$, lying in a closed half-$\mathbb{S}^n$, that is a combinatorial n-simplex, with facet areas $S_1, \ldots, S_{n+1}$. Its faces of any dimension (including the complex itself) have their dihedral angles at most π (thus are convex), but sometimes equal to π.

3. Tools for the Euclidean case: Minkowski's theorems

Now we recall some classical concepts and theorems, in essence due to Minkowski (but in their final form due to A. D. Aleksandrov [1] and W. Fenchel-B. Jessen [20]), cf. also [47], cited below at the respective places. First we do this for arbitrary convex bodies, but then we will give their restrictions to convex polytopes. However, we actually will need general convex bodies when considering convergent sequences of convex polytopes, in our first two proofs of Theorem 2 (Sections 4.3 and 4.4). The third proof only uses Minkowski's Theorem for convex polytopes (Theorem F′). The reader may want to skip directly to Theorem F′.

A *convex body in* $\mathbb{R}^n$ is a compact convex set $K \subset \mathbb{R}^n$ with interior points. For $x \in \partial(K)$ we say that $u \in \mathbb{S}^{n-1}$ is an *outer normal unit vector for K at x*, if $\max\{\langle k, u \rangle \mid k \in K\} = \langle x, u \rangle$. In this section we suppose $n \geq 2$, although later the theorems of this section will be applied only for $n \geq 3$.

Definition 1 (Minkowski, Aleksandrov [1], Fenchel-Jessen [20], cf. also [47, p. 207, (4.2.24) (with $\tau(K, \omega)$ defined on p. 77)]). Let $K \subset \mathbb{R}^n$ be a convex body. The *surface area measure* μ_K *of K* is a finite Borel measure on $\mathbb{S}^{n-1}$, defined in the following way. For a Borel set $B \subset \mathbb{S}^{n-1}$, one defines $\mu_K(B)$ as the $(n-1)$-Hausdorff measure of the set $\{x \in \partial(K) \mid$ there exists an outer normal unit vector u to K at x, such that $u \in B\}$.

Thus μ_K is an element of $C(\mathbb{S}^{d-1})^*$, the dual space of the space of real valued continuous functions $C(\mathbb{S}^{d-1})$ on $\mathbb{S}^{d-1}$, i.e., the finite signed Borel measures on $\mathbb{S}^{d-1}$. We will use the weak* topology of $C(\mathbb{S}^{d-1})^*$, as the topology for the finite (signed) Borel measures μ_K. That is, convergence of a sequence (or, more generally, a net) of finite signed Borel measures $\mu_\alpha \in C(\mathbb{S}^{d-1})^*$ means convergence of each $\int_{\mathbb{S}^{n-1}} f(u) d\mu_\alpha(u)$, where $f \in C(\mathbb{S}^{n-1})$. Moreover, since $\mathbb{S}^{n-1}$ is a compact metric space, the space $C(\mathbb{S}^{n-1})$ is separable, and, hence, the weak* topology of $C(\mathbb{S}^{n-1})^*$ is metrizable. Therefore, it suffices to give the convergent sequences in it (i.e., nets are, in fact, not necessary to be dealt with).

For these elementary concepts and facts from functional analysis, we refer to [16].

Theorem F (Minkowski, Aleksandrov [1], Fenchel-Jessen [20], cf. also [47, p. 389, (7.1.1), pp. 389–390, p. 392, Theorem 7.1.2., p. 397, Theorem 7.2.1]). *Let $n \geq 2$ be an integer, and $K \subset \mathbb{R}^n$ be a convex body. The measure μ_K, defined in Definition 1, is invariant under translations of K and satisfies the following properties.*

   (i) *$\int_{\mathbb{S}^{n-1}} u d\mu_K(u) = 0$, and*
   (ii) *μ_K is not concentrated on any great-$\mathbb{S}^{n-2}$ of $\mathbb{S}^{n-1}$.*

Conversely, for any finite Borel measure μ on $\mathbb{S}^{n-1}$ satisfying (i) *and* (ii)*, there exists a convex body K such that $\mu_K = \mu$. Moreover, this convex body K is unique up to translations.*

Thus, we can consider the map $K \mapsto \mu_K$, also as a map {translates of K} $\mapsto \mu_K$, which we will do later.

Theorem G (Minkowski, Aleksandrov [1], Fenchel-Jessen [20], cf. also [47, p. 198, Theorem 4.1.1, p. 205, pp. 392–393, proof of Theorem 7.1.2]). *Let $n \geq 2$ be an integer. Then the mapping {translates of K} $\mapsto \mu_K$, defined in Definition 1 and just before this theorem, is a homeomorphism between its domain and its range. Its domain is the quotient topology of the topology on the convex bodies induced by the Hausdorff metric, with respect to the equivalence relation of being translates. Its range is the set of finite Borel measures on $\mathbb{S}^{n-1}$ satisfying* (i) *and* (ii) *of Theorem F, with the subspace topology of the weak* topology on $C(\mathbb{S}^{n-1})^*$.*

We have to remark that [47, pp. 392–393, proof of Theorem 7.1.2], as well as [1, proof of the theorem on p. 36, on p. 38], contains explicitly only the proof of the continuity of the bijection {translates of K} $\mapsto \mu_K$, but in fact also the continuity of the inverse map is proved at both places — although not explicitly announced. In fact, as kindly pointed out to the authors by R. Schneider, one has to make the following addition to his book [47, proof of Theorem 7.1.2]. If the sequence of surface area measures μ_{K_i} of some convex bodies $K_i \subset \mathbb{R}^n$ is convergent to the surface area measure μ_K of some convex body $K \subset \mathbb{R}^n$, in the weak* topology, then all K_i's have a bounded diameter — which is announced there for polytopes only, but its proof, given there, is valid for any convex bodies. By a translation, one can achieve that all K_i's, and also K, are contained in a fixed ball — say, let their barycentres be at 0. By compactness of the set of non-empty compact closed sets contained in some closed ball, in the usual topology (i.e., that of the Hausdorff metric), we can choose a convergent subsequence K_{i_j} of K_i, with limit K', say, having as surface area measure $\mu_{K'}$ the weak* limit of the $\mu_{K_{i_j}}$'s, i.e., the originally considered μ_K. By continuity of the barycentre, also the barycentre of K' is 0, as well as that of K; hence $K' = K$. Then the entire sequence K_i converges to K, since else we could choose another subsequence K_{i_k}, converging to another convex body K'', also with barycentre at 0, and with $\mu_{K''} = \mu_K$. This is a contradiction.

Also by Minkowski, a convex body K is a convex polytope if and only if μ_K (that satisfies (i) and (ii) of Theorem F) has a finite support [47, p. 390, Theorem 7.1.1, also considering p. 397, Theorem 7.2.1]. If the support is $\{u_1, \ldots, u_m\}$, then we may write

$$\mu_K = \sum_{i=1}^{m} \mu_K(\{u_i\})\delta(u_i),$$

where $\delta(u_i)$ is the *Dirac measure concentrated at* u_i (i.e., for $B \subset \mathbb{S}^{n-1}$ a Borel set, we have $\delta(u_i)(B) = 0 \iff u_i \notin B$, and $\delta(u_i)(B) = 1 \iff u_i \in B$). Writing such an equation, we always suppose that $\mu_K(\{u_i\}) \neq 0$ for all $i \in \{1, \ldots, m\}$. (Thus the empty sum means the 0 (finite signed Borel) measure; although for a convex body K, we have $\mu_K \neq 0$.) The weak* topology, restricted to finite signed Borel measures of finite support (although we will use only the case when we have a finite Borel measure, and (i) and (ii) of Theorem F hold), where the support has at most m elements, is the following. For $u_\alpha, u \in \mathbb{S}^{d-1}$, with $u_\alpha \to u$, and $c_\alpha, c \in \mathbb{R} \setminus \{0\}$, with $c_\alpha \to c$ (where the u_α's and c_α's are nets indexed by α's from the same index set), we have $c_\alpha \delta(u_\alpha) \to c\delta(u)$, and for $u_\alpha \in \mathbb{S}^{n-1}$ arbitrary, and $c_\alpha \to 0$, we have $c_\alpha \delta(u_\alpha) \to 0$ (thus the convergence is defined for finite signed Borel measures of support having at most one point). Then of course the same convergence holds for finite sums of such sequences (and in fact, only for these, see the formal definition in the next paragraph).

More exactly, a sequence (or, more generally, a net) $\mu_\alpha = \sum_{i=1}^{m_\alpha} \mu_\alpha(\{u_i\})\delta(u_i)$ of finite signed Borel measures on $\mathbb{S}^{d-1}$, with $m_\alpha \leq m$, can converge only to a finite signed Borel measure of support of at most m points. Moreover, it tends to a finite signed Borel measure $\mu = \sum_{i=1}^{m'} \mu(\{u_i\})\delta(u_i)$ on $\mathbb{S}^{n-1}$, with $1 \leq m' \leq m$, if and only if the following holds. For each α there is a partition of $\{1, \ldots, m_\alpha\}$ of cardinality m', say $\{P_{\alpha 1}, \ldots, P_{\alpha m'}\}$ (where each $P_{\alpha j}$ is non-empty), such that

(A) for any $j \in \{1, \ldots, m'\}$, we have that the sets $P_{\alpha j}$ converge to u_j (i.e., for any neighbourhood U_j of u_j, and for all sufficiently large α, we have $P_{\alpha j} \subset U_j$), and

(B) for any $j \in \{1, \ldots, m'\}$, we have that $\sum\{\mu_\alpha(\{u_i\}) \mid i \in P_{\alpha j}\}$ converges to $\mu(\{u_j\})$.

The same sequence (or, more generally, a net) tends to the 0 (finite signed Borel) measure, if and only if

(C) $\sum\{|\mu_\alpha(\{u_i\})| \mid i \in \{1, \ldots, m_\alpha\}\} \to 0$. (This corresponds to the case $m' = 0$, and also here an empty sum means 0.)

For convex polytopes, Theorem F can be rewritten for $\mu_K = \sum_{i=1}^{m} \mu_K(\{u_i\})\delta(u_i)$ as follows.

Theorem F′ (Minkowski, cf. also [47, p. 389, (7.1.1), pp. 389–390, p. 390, Theorem 7.1.1, p. 397, Theorem 7.2.1])**.** *Let $m > n \geq 2$ be integers, let $S_1, \ldots, S_m > 0$, and let $u_1, \ldots, u_m \in \mathbb{S}^{n-1}$. Then there exists a non-degenerate convex polytope P with m facets, with facet areas $S_1, \ldots, S_m$, and with respective facet outer unit normals $u_1, \ldots, u_m$, if and only if*

(i) $\sum_{1=1}^{m} S_i u_i = 0$, *and*

(ii) $u_1, \ldots, u_m$ *do not lie in a linear $(n-1)$-subspace of $\mathbb{R}^n$.*

Moreover, if P exists, it is unique up to translations.

For convex polytopes, with at most m facets, Theorem G can be rewritten for $\mu_K = \sum_{i=1}^{m} \mu_K(\{u_i\})\delta(u_i)$ as follows.

Theorem G′ (Minkowski, cf. also [47, p. 198, Theorem 4.1.1, p. 205, pp. 392–393, proof of Theorem 7.1.2], and the addition after our Theorem G)**.** *Let $m > n \geq 2$ be integers. Then the mapping $\{$translates of $K\} \mapsto \mu_K$, defined in Definition 1, and after Theorem F, is a homeomorphism between its domain and its range. Its*

domain is the subspace corresponding to the non-degenerate convex polytopes with at most m facets, of the quotient topology of the topology on the convex bodies, induced by the Hausdorff metric, with respect to the equivalence relation of being translates. Its range is the set of finite Borel measures on $\mathbb{S}^{n-1}$, with supports of at most m points, satisfying (i) *and* (ii) *of Theorem F', with the subspace topology of the weak* topology on $C(\mathbb{S}^{d-1})^*$. This subspace topology is described in a more explicit form in the last but one paragraph before Theorem F'.*

4. Proofs for the Euclidean case

Essentially the following proposition was used in [27], without explicitly stating and proving it. It can be considered as folklore (as part of the proof of the folklore Theorem 1), but we state and prove it, for completeness.

Proposition 6. *Let $m > n \geq 3$ be integers. For $\delta > 0$ and for numbers $S_m \geq S_{m-1} \geq \ldots \geq S_1 > 0$ such that $S_m < S_1 + S_2 + \ldots + S_{m-1}$, there are pairwise distinct vectors $v_1, v_2, \ldots, v_m \in \mathbb{S}^{n-1}$ that*
(i) *lie in the open δ-neighbourhood of the $x_1 x_2$-coordinate plane, but*
(ii) *do not lie in a linear $(n-1)$-subspace of $\mathbb{R}^n$,*
such that $S_1 v_1 + S_2 v_2 + \ldots + S_m v_m = 0$.

Proof. Consider the standard basis $\{e_i \mid i = 1, \ldots, n\}$ in $\mathbb{R}^n$. Let π be the $x_1 x_2$-coordinate plane.

Since $S_m < S_1 + S_2 + \ldots + S_{m-1}$, there exists a convex polygon $A_1 A_2 \ldots A_m$ in π, such that $|A_i A_{i+1}| = S_i$, for $i = 1, \ldots, m$ (indices considered modulo m); cf. [28, p. 44], [30, pp. 53–54]. We write $\overrightarrow{A_i A_{i+1}} =: |A_i A_{i+1}| u_i$, where $u_i \in \mathbb{S}^{n-1} \cap \pi$ are distinct vectors. Clearly, for $\|v_i - u_i\| < \delta$, the vector v_i lies in the open δ-neighbourhood of π, and also, for δ sufficiently small, the vectors $v_1, \ldots, v_m$ are pairwise distinct.

Let k denote the maximal integer, such that there exist arbitrarily small perturbations $\overrightarrow{B_i B_{i+1}}$ of our original vectors $\overrightarrow{A_i A_{i+1}}$, such that $\sum_{1 \leq i \leq m} \overrightarrow{B_i B_{i+1}} = 0$, and the dimension of the affine hull of $B_1, \ldots, B_m$ has dimension k.

If $k < n$, then, since $m \geq n+1$, there is an affine dependence among the B_i's. Let, e.g., B_m lie in the affine hull of $B_1, \ldots, B_{m-1}$. Then, fixing $|B_{m-1} B_m|$ and $|B_m B_1|$, the point B_m can move on an $(n-2)$-sphere in a hyperplane perpendicular to the above affine hull. Hence, there is an arbitrarily small perturbation of B_m, lying outside of this affine hull, while $\text{aff}\{B_1, \ldots, B_{m-1}\}$ coincides with the affine hull of $B_1, \ldots, B_{m-1}$, and the original B_m. We get a contradiction to the choice of k.

This proves $k = n$, and thus Proposition 6. $\qquad \square$

4.1. **Proof of Theorem 1.** The implication (ii) $\Longrightarrow$ (i) is evident.

To see (i) $\Longrightarrow$ (iii) (which is well-known, but whose proof we include for completeness), using the notations from Theorem F', we have $S_m = \|S_m u_m\| = \|\sum_{i=1}^{m-1} S_i u_i\| \leq \sum_{i=1}^{m-1} S_i$. The only case of equality is the degenerate case as given in (iii') of the theorem.

Lastly, (iii) $\Longrightarrow$ (ii) follows from Proposition 6, and Minkowski's Theorem F'.

The degenerate case, with (iii'), follows from the above considerations. $\qquad \square$

4.2. **Proofs for Theorem 2.** We need the following relation between the surface area, diameter, and volume of a convex body. Here κ_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Proposition 7 (Gritzmann, Wills, and Wrase [23]). *Let $K \subset \mathbb{R}^n$ be a convex body. Then the inequality $S(K)^{n-1} > \kappa_{n-1} \cdot \operatorname{diam}(K) \cdot (nV(K))^{n-2}$ holds, and this inequality is sharp.* $\qquad\square$

We will construct the polytope for Theorem 2 by Minkowski's Theorem F′. We only need to choose an appropriate surface area measure. For a convex polytope, this finite Borel measure is concentrated in finitely many points. If, for given facet areas, we are far from the degenerate case, where this measure would be concentrated in a great-$\mathbb{S}^{n-2}$, then by compactness, the volume of the convex polytope would be bounded from below. Therefore, to get an arbitrarily small volume, we must approach the degenerate case. This will be done in the following proof. Recall also the paragraph after Theorem E, citing [27], where also the degenerate case was approximated — however, only for simplices.

4.3. **First proof of Theorem 2.** We will prove the statement for non-degenerate convex polytopes.

By Proposition 6, for any $\delta > 0$, we find pairwise distinct vectors $v_1, v_2, \ldots, v_m \in \mathbb{S}^{n-1}$ in the open δ-neighbourhood of the x_1x_2-coordinate plane, that do not lie in a linear $(n-1)$-subspace of $\mathbb{R}^n$, such that $S_1v_1 + S_2v_2 + \ldots + S_mv_m = 0$. By Minkowski's Theorem F′, there is a non-degenerate convex polytope $P = P(\delta)$ in $\mathbb{R}^n$, that has m facets with respective areas $S_1, \ldots, S_m$, and with the unit outer normals $v_1, \ldots, v_m$ to the respective facets.

Let us consider the sequence of polytopes $P_k = P(1/k)$, where $k \geq 1$ is an integer. We will show that $V(P_k) \to 0$ as $k \to \infty$. Suppose the contrary. Then, possibly passing to a subsequence, without loss of generality, we may suppose that $V(P_k) \geq \alpha > 0$.

By Proposition 7 we get the inequality

$$S(P_k)^{n-1} > \kappa_{n-1} \cdot \operatorname{diam}(P_k) \cdot (nV(P_k))^{n-2} \geq \kappa_{n-1} \cdot \operatorname{diam}(P_k) \cdot (n\alpha)^{n-2},$$

where $S(P_k) = \sum_{i=1}^m S_i$ is a constant. From this inequality, we get that $\operatorname{diam}(P_k)$ is bounded by some constant D for every k.

Further, by applying some translations, we may suppose without loss of generality, that all P_k's have a common point. Therefore, all polytopes P_k lie in a ball of radius D. Using compactness, possibly passing to a subsequence, we may suppose even more: the sequence P_k tends (in the Hausdorff metric) to a non-empty compact convex set P_0 as $k \to \infty$ [47, p. 50, Theorem 1.8.6]. Since volume is continuous for non-empty compact convex sets [47, p. 55, Theorem 1.8.16], $V(P_0) \geq \alpha > 0$, and hence P_0 is a convex body. Moreover, by Minkowski's Theorem G′ (actually only by the continuity of the bijection in that theorem), P_0 is a convex polytope, with m facets, and facet areas $S_1, \ldots, S_m$, and facet outer unit normals in the x_1x_2-coordinate plane. However, this is a contradiction to Theorem F′, (ii). $\qquad\square$

Remark 6. In fact, we could have used, rather than Proposition 7, in which the multiplicative constant is sharp, a consequence of the Aleksandrov-Fenchel inequality [47, p. 327, Theorem 6.3.1]. Namely: the quermassintegrals $W_i(K)$ [47, p. 209], for $0 \leq i \leq n$, form a logarithmically concave sequence, where $W_0(K) = V(K)$, and, for

n fixed, $W_1(K)$ is proportional to $S(K)$, and $W_{n-1}(K)$ is proportional to the mean width of K [47, p. 210, p. 291, (5.3.12)]. Then apply this logarithmic convexity for volume, constant times surface area, and constant times mean width, noting that the quotient of the diameter and the mean width is between two positive numbers (only depending on n).

Example 2. We give an example of a family of tetrahedra ($n = 3$ and $m = 4$) with constant facet areas and arbitrarily small volume. The tetrahedron looks like a thin vertical needle and has vertices $(\pm\varepsilon, 0, -(1/\varepsilon)\sqrt{1 - \varepsilon^4/4})$ and $(0, \pm\varepsilon, (1/\varepsilon)$ $\sqrt{1 - \varepsilon^4/4})$. All facets have area 2, and the volume is $(4\varepsilon/3)\sqrt{1 - \varepsilon^4/4}$, which goes to zero as $\varepsilon \to 0$.

4.4. **Second proof of Theorem 2.** We will prove the statement for non-degenerate convex polytopes.

1. First, we will construct a partition $\mathcal{P} = \{P_1, \ldots, P_{n+1}\}$ of the indices $i \in \{1, \ldots, m\}$ into $n+1$ classes. We will achieve that the $n+1$ numbers $\sum_{i \in P_j} S_i$ (for $1 \leq j \leq n+1$) also satisfy that

      (∗) the largest of these numbers is smaller than the sum of all others.

We start with the partition with all classes having one element. Suppose that we already have constructed a partition $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$, such that

      (∗∗) the largest of the numbers $\sum_{i \in Q_j} S_i$, where $1 \leq j \leq k$, is smaller than
           the sum of all other of these numbers.

If $k = n+1$, we stop. If $k > n+1$, let us denote the partition sums by $T_j := \sum_{i \in Q_j} S_i$, and assume $T_1 \leq T_2 \leq \ldots \leq T_k$. Now we take the two classes Q_1 and Q_2 with the two smallest sums and form their union, while the other classes Q_j are retained. In the new partition, the partition class having maximal sum T_j can be either the same partition class as in the preceding step, or the new constructed union. In the first case, (∗∗) is evident. In the second case, we have $T_1 + T_2 \leq T_{k-1} + T_k$ before taking the union, since $k \geq n+2 \geq 5$, and thus $T_{\text{new}} := T_1 + T_2 < T_3 + \ldots + T_{k-1} + T_k$. So, (∗∗) is once more satisfied.

This proves that for the final partition we in fact have (∗).

2. Now we consider the partition $\mathcal{P} = \{P_1, \ldots, P_{n+1}\}$ constructed above. For the sums $R_j := \sum\limits_{i \in P_j} S_i$, we can construct a non-degenerate simplex S, having these facet areas, and volume arbitrarily small, by Theorem E. Then, for the respective outer unit normals u_j of the facets of this simplex, we have $\sum_{j=1}^{n+1} R_j u_j = 0$.

3. Let $\varepsilon > 0$ be small. For each $1 \leq j \leq n+1$, and each $i \in P_j$, we find vectors u_{ji}, in some linear 2-subspace X_j, containing u_j, such that $-(1-\varepsilon)(\sum\limits_{i \in P_j} S_i)u_j, S_i u_{ji}$ are the side vectors of a convex polygon in X_j. Hence all u_{ji} are close to u_j, for ε sufficiently small. Then $S_i u_{ji}$, for all $1 \leq j \leq n+1$ and all $i \in P_j$, satisfy that they linearly span $\mathbb{R}^n$, and their sum is

$$\sum_{1 \leq j \leq n+1} \left(\sum_{i \in P_j} S_i u_{ji}\right) = \sum_{1 \leq j \leq n+1} (1-\varepsilon)\left(\sum_{i \in P_j} S_i\right) u_j = (1-\varepsilon) \sum_{1 \leq j \leq n+1} R_j u_j = 0.$$

4. By Minkowski's Theorem F′, there exists a non-degenerate convex polytope with facet outer unit normals u_{ji}, and facet areas S_i.

Since we have changed in the course of the proof the surface area measure only a little bit (in the weak*-topology of $C(\mathbb{S}^{n-1})^*$), the obtained convex polytope is arbitrarily close to the original simplex S, after a suitable translation, by Theorem G′ (actually only by the continuity of the inverse of the bijection in that theorem). Since the simplex had an arbitrarily small volume, by continuity of the volume for convex bodies, our convex polytope also has an arbitrarily small volume. □

4.5. **Third proof of Theorem 2.** We will prove the statement for non-degenerate convex polytopes.

The first two proofs of Theorem 2 did not give geometric informations about the constructed polytopes. (The first proof used an argument by contradiction, and the second proof used the examples of the simplices.) Now we give a third proof, which is more quantitative and will give also geometric information: the examples given in that proof are "needle-like", similarly to Example 2. Cf. also the paragraph following the statement of Theorem 2.

First we give the proof for $n = 3$ dimensions.

We begin with an elementary lemma. It shows that convex polyhedra in $\mathbb{R}^3$, with steep (almost vertical) facets, and with angles of outer normal unit vectors of different facets bounded away from 0 and π, have steep edges.

Lemma 1. *Consider two planes in $\mathbb{R}^3$, whose unit normals u^+ and u^- enclose an angle at most $\varepsilon \in (0, \pi/2)$ with the xy-plane, and whose angle with each other lies in $[\beta, \pi - \beta]$, where $0 < \beta \le \pi/2$. Then their intersection line encloses an angle at most*

$$\delta := \arcsin \frac{\sin \varepsilon}{\sin(\beta/2)}$$

with the z-axis, provided that $\varepsilon \le \beta/2$. This inequality is sharp.

Proof. We choose a new coordinate system where the intersection line becomes the vertical axis, and the two normal vectors $u^+, u^- \in \mathbb{S}^2$ lie in the horizontal plane, making an angle $\beta' \in [\beta, \pi - \beta]$ with each other. The intersection line of the two planes is $\lim \{(0, 0, 1)\}$. At the same time, in the new coordinate system, the North Pole becomes $n = (n_1, n_2, n_3) \in \mathbb{S}^2$.

By hypothesis,

$$\langle n, u^- \rangle, \langle n, u^+ \rangle \in [-\sin \varepsilon, \sin \varepsilon]. \tag{6}$$

We want to conclude that

$$|\langle (0, 0, 1), n \rangle| = |n_3| \ge \cos \delta, \tag{7}$$

i.e., that

$$\sqrt{n_1^2 + n_2^2} \le \sin \delta. \tag{8}$$

The points $(n_1, n_2) \in \mathbb{R}^2$ (projections of n to the xy-plane) for n satisfying (6), form a rhomb of heights $2 \sin \varepsilon$ and angles $\beta', \pi - \beta'$. A farthest point of this rhomb from $(0, 0)$ must be one of the vertices, and its distance from $(0, 0)$ is $\max\{(\sin \varepsilon)/\sin(\beta'/2),$ $(\sin \varepsilon)/\cos(\beta'/2)\} \le (\sin \varepsilon)/\sin(\beta/2) = \sin \delta$. That is, (8), or equivalently, (7), holds, both being sharp inequalities. Hence, the inequality of the lemma holds, and is sharp. □

Lemma 2. *Consider a convex polyhedron $P \subset \mathbb{R}^3$ with facet areas $S_1, \ldots, S_m$. Suppose that its facet outer normals enclose an angle at most ε with the xy-plane, and their pairwise enclosed angles lie in $[\beta, \pi - \beta]$, where $0 < \beta \leq \pi/2$. Then its volume is bounded by*

$$V(P) \leq 2^{-1/4}\pi^{-1} \cdot \left(\sum_{i=1}^{m} S_i^{3/4}\right)^2 \cdot \left(\frac{\sin\varepsilon}{\sin(\beta/2)}\right)^{1/2},$$

if $(\sin\varepsilon)/\sin(\beta/2) \leq 1/\sqrt{2}$.

Proof. We denote by $s_i(z)$ the length of the horizontal cross-section of the i-th facet at height z, and by $s_i^{\max}$ the maximum length of such a horizontal cross-section. Let h_i be the "height" of the i-th face: the difference between the maximum and the minimum z-coordinates of its points. Let h_i' be the "tilted height" of this facet in its own plane, i.e., the height when the plane is rotated into vertical position about one of its horizontal cross-sections.

Since $(\sin\varepsilon)/\sin(\beta/2) < 1$, we have by Lemma 1 that P has no horizontal edges. Therefore, once more by Lemma 1, we get

$$s_i^{\max} \leq h_i \cdot \tan\delta,$$

because, from the minimum z-coordinate, where $s_i(z) = 0$, we have that $s_i(z)$ can only increase with speed at most $2\tan\delta$ ($< \infty$) to reach its maximum $s_i^{\max}$ (this is clear for a vertical face, and, for a nonvertical face, the speed is even smaller: observe that the i'th facet lies in an upwards circular cone, with vertex the lowest point of the i'th facet, and directrices encosing an angle δ with the z-axis), and from there it must decrease again, with speed at most $2\tan\delta$, till 0 at the maximum z-coordinate.

Therefore,

$$S_i \geq s_i^{\max} h_i'/2 \geq s_i^{\max} h_i/2 \tag{9}$$
$$\geq (s_i^{\max})^2/(2\tan\delta),$$

which gives

$$s_i^{\max} \leq \sqrt{2 S_i \tan\delta}. \tag{10}$$

These relations allow us to bound the volume $V(P)$ as follows, after using the isoperimetric inequality on each horizontal slice:

$$V(P) = \int_{-\infty}^{\infty} (\text{area of cross-section of } P \text{ at height } z) \, dz$$

$$\leq \int_{-\infty}^{\infty} \left(\sum_{i=1}^{m} s_i(z)\right)^2 dz/(4\pi)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \int_{-\infty}^{\infty} s_i(z)s_j(z) \, dz/(4\pi)$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{m} s_i^{\max} s_j^{\max} \min\{h_i, h_j\}/(4\pi) \tag{11}$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{m} s_i^{\max} s_j^{\max} \sqrt{h_i h_j} / (4\pi)$$

$$= \left(\sum_{i=1}^{m} s_i^{\max} \sqrt{h_i} \right)^2 / (4\pi)$$

$$= \left(\sum_{i=1}^{m} \sqrt{s_i^{\max}} \sqrt{s_i^{\max} h_i} \right)^2 / (4\pi)$$

$$\leq \left(\sum_{i=1}^{m} (2 S_i \tan \delta)^{1/4} \sqrt{2 S_i} \right)^2 / (4\pi) \tag{12}$$

$$= \frac{\left(\sum_{i=1}^{m} S_i^{3/4} \right)^2}{\sqrt{2}\pi} \cdot \sqrt{ \frac{(\sin \varepsilon) / \sin(\beta/2)}{\sqrt{1 - (\sin^2 \varepsilon) / \sin^2(\beta/2)}} } \tag{13}$$

$$\leq \frac{\left(\sum_{i=1}^{m} S_i^{3/4} \right)^2}{\sqrt{2}\pi} \cdot \sqrt{ \frac{\sqrt{2} \sin \varepsilon}{\sin(\beta/2)} }.$$

The first inequality uses the isoperimetric inequality, and the second inequality (11) simply bounds the integral by an upper bound of the non-negative integrand times the length of the interval where the integrand is positive. For (12), we have used (9) and (10); to obtain (13), we used Lemma 1, and the last inequality simplifies the denominator by the assumption $(\sin \varepsilon) / \sin(\beta/2) \leq 1/\sqrt{2}$ of the lemma. $\square$

4.6. **Third proof of Theorem 2 for $n = 3$ dimensions.** Like in the first proof, we use Minkowski's Theorem F′. We want to apply Lemma 2, making ε small. Thus, we must let the normal vectors with given lengths S_i converge to the xy-plane, keeping their sum to be 0, with the linear span of the facet outer unit normals being $\mathbb{R}^3$, and then apply Minkowski's Theorem F′. In the limiting configuration, the normals will lie in the xy-plane. They must form angles in $[\beta, \pi - \beta]$, for $\beta \in (0, \pi/2]$, with each other, for Lemma 2 to work. Thus we must avoid parallel sides.

Consider a convex m-gon M with sides S_i, having the minimum number of parallel pairs of sides. Let us suppose that M has a side such that the angles at its vertices have a sum different from π. Then, by a small motion of this side and the neighbouring two sides, one can achieve that this side changes its direction, while new parallel pairs of sides are not created. Therefore, M can have a parallel pair of sides only if for both of these sides the sums of the angles at their vertices are π. That is, we have four vertices, determining two parallel sides, whose outer angles (i.e., π minus the inner angles) have sum 2π. Since the sum of all outer angles is 2π, there are no more vertices, and M must be a parallelogram. If its sides are not equal, we rearrange the side vectors so as to obtain a (convex) deltoid, which is not a parallelogram. So there remains the case when $m = 4$ and $S_1 = S_2 = S_3 = S_4$. This case, however, has been treated by Example 2, a tetrahedron with four equal faces. Suitable inflations provide examples for all values of S_i.

Now we have a strictly convex polygon in the xy-plane, without parallel sides. Suppose the angle between any two edges is in the range $[\beta_1, \pi - \beta_1]$, for some $\beta_1 > 0$. We still need to perturb the sides such that the edge vectors span $\mathbb{R}^3$.

Consider the first three consecutive vertices $A_1A_2A_3$ of M and rotate the two sides A_1A_2 and A_2A_3 about the line through A_1 and A_3 through a small angle $\alpha > 0$, keeping their lengths fixed. The $m - 2 \geq 2$ remaining sides span the xy-plane, since they are not parallel; the vector A_1A_2 points out of the xy-plane, and therefore the edge vectors span $\mathbb{R}^3$.

By making the angle of rotation α small enough, we can ensure that the angle between all edge vectors of the perturbed polygon $M(\alpha)$ is still in the range $[\beta_2, \pi - \beta_2]$, for some fixed $\beta_2 > 0$, and the angle ε with the xy-plane can be made as small as we want. We use the edge vectors $\overrightarrow{S_i}$ of $M(\alpha)$ as outer normals and construct the polytope P by Minkowski's Theorem F$'$ (with $S_i = \|\overrightarrow{S_i}\|$, and $u_i = \overrightarrow{S_i}/S_i$). By Lemma 2, the volume can be made as small as we want. $\qquad\square$

In the polytope P that we have constructed, all facets except two are vertical. By going through the proof of Lemma 2, one can see that it would therefore have been sufficient to enforce the constraint $[\beta, \pi - \beta]$ on the angles for those pairs of facet normals that involve one of the two nonvertical facets.

Example 3. For odd dimension $n = 2k + 1$, there is a higher-dimensional generalization of Example 2. Consider a (large) regular k-simplex of edge length $a := 1/\varepsilon$ in the $x_{k+2} \ldots x_d$-coordinate plane. It has $k + 1$ vertices $v_1, \ldots, v_{k+1}$. At each vertex v_i, we draw a short segment of length $b := \varepsilon$, centred at v_i, in the direction of the x_i-axis. The convex hull of the union of these segments is a d-simplex with congruent facets. The facet areas are $\sim \text{const} \cdot a^k b^k = \text{const}$, while the volume is $\text{const} \cdot a^k b^{k+1} = \text{const} \cdot \varepsilon$, becoming arbitrarily small as $\varepsilon \to 0$.

4.7. Third proof of Theorem 2, for n > 3 dimensions.

Like for $n = 3$, we have a planar convex m-gon M in the x_1x_2-coordinate plane, with side vectors $\overrightarrow{S_i}$ (this notation will be preserved also after perturbations), and with side lengths S_i, where $1 \leq i \leq m$ (and $m \geq n + 1$).

Let us consider the $x_1 \ldots x_{n-1}$-coordinate hyperplane X, which contains M. By small generic perturbations of the closed polygon M in X, preserving the side lengths S_i, we want to achieve that

$(*)$ the perturbed polygon $M \subset X$ has no $n - 1$ side vectors lying in an at most $(n - 2)$-dimensional linear subspace of X.

Initially, M lies in a 2-dimensional plane. We will fulfill $(*)$ by following the proof of Proposition 6. (Our desired conclusion is slightly stronger than in Proposition 6; there we only excluded the case that *all* vectors lie in a lower-dimensional subspace.)

Assume that some $i \leq n - 1$ side vectors lie in a linear subspace of dimension less than i, where i is the smallest such number. We will eliminate these linear dependencies iteratively. Let us select the smallest such i. We have already seen how we can avoid parallel edges, and therefore we can assume $i \geq 3$. (The only case where parallel sides could not be avoided was $m = 4$ and $S_1 = S_2 = S_3 = S_4$ (a rhomb), and this happens only for $n \leq m - 1 = 3$.)

Observe that we may any time rearrange the cyclic order of side vectors of M as we want. So we may assume that any $i - 1$ side vectors are linearly independent, and some i linearly dependent side vectors are $\overrightarrow{S_1}, \ldots, \overrightarrow{S_i}$. Number the vertices A_k so that $\overrightarrow{S_k}$ goes from vertex A_k to A_{k+1} (indices meant cyclically). Now, fixing A_1, A_3, $\|\overrightarrow{S_1}\|$ and $\|\overrightarrow{S_2}\|$, the point A_2 can move on an $(n - 3)$-dimensional

sphere in a hyperplane within X, with affine hull orthogonal to the segment $[A_1, A_3]$. There is a small generic motion that moves A_2 out of the $(i-1)$-dimensional subspace $\mathrm{aff}\{A_1, A_3, \ldots, A_{i+1}\}$. Then $\dim \mathrm{aff}\{A_1, A_2, \ldots, A_i, A_{i+1}\})$ increases by 1, and $\overrightarrow{S_1}, \ldots, \overrightarrow{S_i}$ become linearly independent. If the perturbation is small enough (or sufficiently generic), every set of side vectors that was linearly independent before the motion remains linearly independent. Therefore, the number of linearly dependent i-tuples of side vectors of M decreases. A finite number of iterations eliminates all linearly dependent i-tuples. We can then increase i and continue this process until i becomes n, and $(*)$ is established.

Condition $(*)$ can be rephrased as saying that the determinant of any $n-1$ normed side vectors $\overrightarrow{S_i}/S_i$ of M (i.e., the signed volume of the parallelepiped spanned by them) is nonzero. We denote by $b > 0$ the smallest absolute value of all of these determinants. This bound will play the role of the sine of the angle bound β in Lemmas 1 and 2.

We have the following generalization of Lemma 1.

Lemma 3. *Let $n \geq 3$, and consider $n-1$ hyperplanes in $\mathbb{R}^n$, making an angle at most $\varepsilon < \pi/2$ with the vertical axis (the x_n-axis). If their unit normal vectors span an $(n-1)$-parallelotope of volume at least b (> 0), then they intersect in a line. The angle between this line and the vertical direction is bounded by*

$$\delta := \arcsin \frac{(n-1)^{3/2} \sin \varepsilon}{b}$$

provided that $(n-1)^{3/2} \sin \varepsilon \leq b$.

For fixed n, the order of magnitude of this bound on δ as a function of ε and b is optimal. More precisely, for any ε and b (where $0 < \varepsilon < \pi/2$ and $0 < b \leq 1$), there are instances with $\sin \delta = \min\{1, (\sin \varepsilon)/\sin((\arcsin b)/2)\}$.

Proof. Since the unit normal vectors $v_1, \ldots, v_{n-1}$ are linearly independent, the intersection of the hyperplanes is a line ℓ. Let us choose a new orthonormal coordinate system where ℓ is the last coordinate axis. Then the last coordinate of the vectors v_i is zero, and we may write them as $v_i = \binom{v_i'}{0}$ with $v_i' \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$. By assumption, the $(n-1) \times (n-1)$ matrix $V = (v_1', \ldots, v_{n-1}')$ has determinant of absolute value $|\det V| \geq b$.

Let $p = \binom{p'}{p_n}$, with $p' \in \mathbb{R}^{n-1}$, be the unit vector of the original positive x_n-direction in the new coordinate system. Its angle δ with the line ℓ satisfies $\cos \delta = |p_n|$ and $\sin \delta = \|p'\|$, and thus our goal is to show that

$$\|p'\| \leq (n-1)^{3/2} \frac{\sin \varepsilon}{b}. \tag{14}$$

Let α_i denote the angle between p and the normal v_i. By the angle assumption on the hyperplanes, we have $\pi/2 - \varepsilon \leq \alpha_i \leq \pi/2 + \varepsilon$, and thus, with $r_i := \cos \alpha_i = \langle p, v_i \rangle = \langle p', v_i' \rangle$, we have $|r_i| \leq \sin \varepsilon$.

The $n-1$ equations $\langle p', v_i' \rangle = r_i$ form a linear system $(p')^T V = (r_1, \ldots, r_{n-1})$ (the column vectors of V being the v_i's), i.e., $V^T p' = (r_1, \ldots, r_{n-1})^T$, which determines p' uniquely:

$$p' = (V^T)^{-1} (r_1, \ldots, r_{n-1})^T. \tag{15}$$

By the formula $(V^T)^{-1} = \mathrm{adj}\, V^T / \det V^T$ (where $\mathrm{adj}\, V^T$ is the transpose of the matrix with entries the signed cofactors of the respective entries of V^T), each entry

of $(V^T)^{-1}$ is an $(n-2) \times (n-2)$ subdeterminant of V^T, divided by $\pm \det V^T$. The rows of the submatrices of V^T are vectors of length at most 1, and therefore these subdeterminants are bounded in absolute value by 1. It follows that the entries of $(V^T)^{-1}$ are bounded in absolute value by $1/b$. Since the $|r_i|$'s are at most $\sin \varepsilon$, we get from (15) that the $n-1$ entries of p' are bounded by $(n-1)(\sin \varepsilon)/b$ in absolute value, and hence we have proved (14).

To establish the lower bound, we can lift the tight 3-dimensional example from Lemma 1 to n dimensions. The enclosed angle will be the same as in 3 dimensions, namely $\arcsin((\sin \varepsilon)/\sin(\beta/2))$, where $\sin \beta = b$, provided $\varepsilon \leq \beta/2$, while for $\varepsilon > \beta/2$ we use the example with $\varepsilon = \beta/2$. We embed the 3-dimensional example into $\mathbb{R}^n$ by a linear isometry mapping the positive x, y, z-axes to the positive x_1, x_2, x_n-coordinate axes of $\mathbb{R}^n$. (The "vertical" direction is now the direction of the x_n-axis.) The two 2-planes of the three-dimensional example are turned into hyperplanes by replacing them by their inverse images under the orthogonal projection of $\mathbb{R}^n$ to the $x_1 x_2 x_n$-coordinate subspace. Simultaneously, we add the hyperplanes with equations $x_3 = 0, \ldots, x_{n-1} = 0$. $\qquad \square$

Lemma 4. *Let $n > 3$ be an integer. Suppose that a convex polytope $P \subset \mathbb{R}^n$ has facet areas $S_1, \ldots, S_m$, and its facet outward unit normals enclose an angle at most $\varepsilon \in (0, \pi/2)$ with the $x_1 \ldots x_{n-1}$-plane, and the volume of the $(n-1)$-parallelepiped spanned by any $n-1$ unit facet normals of P is at least $b\ (> 0)$. Then its volume is bounded by*

$$V(P) \leq \mathrm{const}_n \cdot \Big(\sum_{i=1}^m S_i^{n/(2n-2)} \Big)^2 \cdot \Big(\frac{\sin \varepsilon}{b} \Big)^{1/(n-1)},$$

if $\sin^2 \varepsilon \leq b^2/[2(n-1)^3]$.

On the other hand, for $n \geq 3$, and any $m \geq 2n$, there is a suitable $\varepsilon_0 \in (0, \pi/4)$, such that the following holds. For any $\varepsilon \in (0, \varepsilon_0)$, there exists a convex polytope $P(\varepsilon) \subset \mathbb{R}^n$, with m facets, which satisfies the hypotheses of this lemma (except the one about the facet areas), with b only depending on n and m, such that

$$\begin{cases} V(P(\varepsilon)) \geq \mathrm{const}'_n \cdot S(P(\varepsilon))^{n/(n-1)} \cdot (\tan \varepsilon)^{1/(n-1)} \\ \geq \mathrm{const}'_n \cdot m^{-(n-2)/(n-1)} \cdot \big(\sum_{i=1}^m S_i(\varepsilon)^{n/(2n-2)} \big)^2 \cdot (\tan \varepsilon)^{1/(n-1)}. \end{cases}$$

(here $S_1(\varepsilon), \ldots, S_m(\varepsilon)$ are the areas of the facets of $P(\varepsilon)$). In particular, in the inequalities of Lemma 2 and this lemma, the order of magnitude, as a function of ε, is optimal.

Proof. We begin with the proof of the upper estimate. We denote by $s_i(x_n)$ the $(n-2)$-volume of the horizontal cross-section of the i-th facet at height x_n, and by $s_i^{\max}$ the maximum $(n-2)$-volume of such a horizontal cross-section. Let h_i be the "height" of the i-th facet: the difference between the maximum and the minimum x_n-coordinates of its points. Let h'_i be the "tilted height" of this facet in its own hyperplane, i.e., the height when the hyperplane is rotated into vertical position about one of its horizontal cross-sections.

Now, since $\sin^2 \varepsilon \leq b^2/[2(n-1)^3]$, the angle δ from Lemma 3 lies in $(0, \pi/2)$. Hence, by Lemma 3, P has no horizontal edges, thus also no horizontal k-faces, for any $k \in \{1, \ldots, n-2\}$. Therefore, once more by Lemma 3, we know that every facet is included in two rotationally symmetric cones, with bases $(n-1)$-balls. One

cone has its apex at the unique lowest point of this facet and extends upwards from there. Its axis is vertical (parallel to the x_n-direction), and the directrices enclose an angle δ with the x_n-axis. The other cone extends downwards from the highest point of the facet, has a vertical axis, and directrices enclosing an angle δ with the x_n-axis. We use the upwards cone from the minimum height till the arithmetic mean of the minimum and maximum heights, and the downward cone for the other half of the vertical extent of the facet. By this argument, we can bound the maximum cross-section area $s_i^{\max}$ of the i-th facet as follows:

$$s_i^{\max} \leq ((h_i/2) \cdot \tan \delta)^{n-2} \cdot \kappa_{n-2}. \tag{16}$$

(From the minimum height till the arithmetic mean of the minimum and maximum heights, any horizontal cross-section of the cone is contained in some $(n-1)$-ball of radius at most $R := (h_i/2) \cdot \tan \delta$. Thus any horizontal cross-section of the facet lies inside the intersection of its own affine hull, that is an $(n-2)$-dimensional affine subspace, with the upwards cone having a base an $(n-1)$-ball of radius R, hence inside some $(n-2)$-ball of radius at most R. A similar argument holds for the downward cone.) Moreover, we also have

$$S_i \geq s_i^{\max} h_i'/(n-1) \geq s_i^{\max} h_i/(n-1). \tag{17}$$

Let us rewrite (16) and (17) as follows:

$$h_i^{-(n-2)} \cdot s_i^{\max} \leq ((\tan \delta)/2)^{n-2} \cdot \kappa_{n-2}, \tag{18}$$

$$h_i \cdot s_i^{\max} \leq (n-1)S_i \tag{19}$$

We multiply the $1/[(2n-2)(n-2)]$-th power of (18) with the $n/(2n-2)$-th power of (19) to get an inequality that we will need:

$$\begin{cases} (s_i^{\max})^{(n-1)/(2n-4)}\sqrt{h_i} \leq \\ ((\tan \delta)/2)^{1/(2n-2)} \cdot (\kappa_{n-2})^{1/[(2n-2)(n-2)]} \cdot ((n-1)S_i)^{n/(2n-2)}. \end{cases} \tag{20}$$

Let $K := [(n-1)^{n-1}\kappa_{n-1}]^{-1/(n-2)}$ denote the constant of the isoperimetric inequality in $n-1$ dimensions:

$$V_{n-1}(C) \leq K \cdot (V_{n-2}(\partial C))^{(n-1)/(n-2)} \tag{21}$$

(for $C \subset \mathbb{R}^{n-1}$). We can now bound the volume as follows.

$$V(P) = \int_{-\infty}^{\infty} \left[(n-1)\text{-volume of the cross-section of } P \text{ at height } x_n\right] dx_n$$

$$\leq \int_{-\infty}^{\infty} \left[\left(\sum_{i=1}^{m} s_i(x_n)\right)^{(n-1)/(2n-4)}\right]^2 dx_n \cdot K$$

$$\leq \int_{-\infty}^{\infty} \left[\sum_{i=1}^{m} s_i(x_n)^{(n-1)/(2n-4)}\right]^2 dx_n \cdot K$$

$$= \int_{-\infty}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} s_i(x_n)^{(n-1)/(2n-4)} s_j(x_n)^{(n-1)/(2n-4)} dx_n \cdot K$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-\infty}^{\infty} s_i(x_n)^{(n-1)/(2n-4)} s_j(x_n)^{(n-1)/(2n-4)} dx_n \cdot K$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{m} (s_i^{\max})^{(n-1)/(2n-4)} (s_j^{\max})^{(n-1)/(2n-4)} \min\{h_i, h_j\} \cdot K \qquad (22)$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{m} (s_i^{\max})^{(n-1)/(2n-4)} (s_j^{\max})^{(n-1)/(2n-4)} \sqrt{h_i h_j} \cdot K$$

$$\leq (\tan \delta)^{1/(n-1)} \cdot 2^{-1/(n-1)} \cdot (\kappa_{n-2})^{1/[(n-1)(n-2)]} \cdot (n-1)^{n/(n-1)} \qquad (23)$$

$$\cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)} \right)^2 \cdot K$$

$$= \operatorname{const}_n \cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)} \right)^2 \left(\frac{(n-1)^{3/2}(\sin \varepsilon)/b}{\sqrt{1 - (n-1)^3 (\sin^2 \varepsilon)/b^2}} \right)^{1/(n-1)} \qquad (24)$$

$$\leq \operatorname{const}'_n \cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)} \right)^2 \cdot \left(\frac{\sin \varepsilon}{b} \right)^{1/(n-1)}$$

The first inequality uses the isoperimetric inequality (21), the second inequality uses concavity of the function $t^{(n-1)/(2n-4)}$ for $t \in [0, \infty)$ and its vanishing at $t = 0$ (observe that $0 < (n-1)/(2n-4) \leq 1$). Inequality (22), as in (11), bounds the integral of a non-negative function by an upper bound of the integrand times the length of the interval where the integrand is positive. For (23), we have used the bound (20) that we derived above. Inequality (24) uses the bound δ from Lemma 3. Finally, by hypothesis, the expression under the square root in the denominator of (24) is bounded below by $1 - (n-1)^3 (\sin^2 \varepsilon)/b^2 \geq 1/2$. We have therefore established the claimed upper bound.

Now we give the example for the lower bound, for $n \geq 3$ and $m \geq 2n$. Let $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0, \pi/4)$ will be chosen later. Let us write $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$. Let $T^+, T^- \subset \mathbb{R}^{n-1}$ be regular $(n-1)$-simplices circumscribed about the unit ball B^{n-1} of $\mathbb{R}^{n-1}$, which are in such a generic position w.r.t. each other, that their altogether $2n$ facet outer unit normals satisfy the following genericity condition: any $n-1$ of these facet outer unit normals linearly span $\mathbb{R}^{n-1}$. Let $n \leq m^+, m^-$, and $m = m^+ + m^-$. Let $R^\pm$ be obtained from $T^\pm$ by intersecting it still with $m^\pm - n$ closed halfspaces in $\mathbb{R}^{n-1}$, each containing B^{n-1}, with its boundary touching B^{n-1}. Then

$$B^{n-1} \subset R^\pm \subset T^\pm \subset (n-1) B^{n-1}.$$

Let the altogether $m = m^+ + m^-$ facet outer unit normals of R^+ and R^- satisfy the same genericity condition as above: any $n-1$ of these facet outer unit normals linearly span $\mathbb{R}^{n-1}$. Let $b > 0$ be the minimum of the $(n-1)$-volumes of the $(n-1)$-parallelotopes spanned by any $n-1$ facet outer unit normals out of these altogether m facet outer unit normals of R^+ and R^-. (Observe that for $n = 3$ and $m \geq 2$, the largest value of b is $\sin(\pi/m)$ — if we do not begin the construction with two regular triangles, but allow any m facet outer unit normals in $\mathbb{S}^{n-2} = \mathbb{S}^1$. For $n > 3$, the maximal value of b can be bounded from above as follows — again not beginning with two regular simplices, but allowing any m facet outer unit normals in $\mathbb{S}^{n-2}$. Let us choose altogether $n-1$ outer normal unit vectors of R^+ and R^-,

say, $u_1, \ldots, u_{n-1} \in \mathbb{S}^{n-2}$. We have

$$
\begin{cases}
|\det (u_1, \ldots, u_{n-1})|/(n-1)! = \\
V_{n-2}(\text{conv} \{u_1, \ldots, u_{n-1}\}) \cdot \text{dist} (0, \text{aff} \{u_1, \ldots, u_{n-1}\})/(n-1) \leq \\
V_{n-2}(\text{conv} \{u_1, \ldots, u_{n-1}\})/(n-1)
\end{cases}
$$

(here dist denotes distance). Thus it suffices to bound $V_{n-2}(\text{conv} \{u_1, \ldots, u_{n-1}\})$ from above. This is the spherical analogue — for $\mathbb{S}^{n-2}$ — of the celebrated Heilbronn problem, which asks for the maximum of the minimal n-volume of n-simplices spanned by any m points in $[0,1]^n$, which problem is poorly understood; for an extensive literature on this problem, cf. [12], Ch. 11.2. Unfortunately this spherical variant cannot be reduced to the case of $[0,1]^{n-2}$, by taking projections of, say, the intersections of all orthants by $\mathbb{S}^{n-2}$, to the tangent $\mathbb{R}^{n-2}$'s at their centres, since the area of conv $\{u_1, \ldots, u_{n-1}\}$ can be large, even if its projection has a small area. In one direction we have an implication: large projection areas imply large areas — however, then large areas still do not imply large values of $|\det (u_1, \ldots, u_{n-1})|$. However, this spherical variant is a special case of the Heilbronn problem for $[0,1]^{n-1}$: namely we can add to any set of $n-1$ vectors in $\mathbb{S}^{n-2}$ still the single vector 0 — but seemingly this way we loose a part of the information.) Let $P^{\pm}(\varepsilon)$ be the half-infinite pyramid with vertex $(0, \ldots, 0, \pm \tan \varepsilon)$, and base $R^{\pm}$. Then

$$
C_i^{\pm}(\varepsilon) \subset P^{\pm}(\varepsilon) \subset C_o^{\pm}(\varepsilon),
$$

where $C_i^{\pm}(\varepsilon)$ or $C_o^{\pm}(\varepsilon)$ is a half-infinite cone with vertex $(0, \ldots, 0, \pm \tan \varepsilon)$, and base B^{n-1}, or $(n-1)B^{n-1}$, respectively. Therefore

$$
C_i(\varepsilon) := C_i^{+}(\varepsilon) \cap C_i^{-}(\varepsilon) \subset P(\varepsilon) := P^{+}(\varepsilon) \cap P^{-}(\varepsilon) \subset C_o(\varepsilon) := C_o^{+}(\varepsilon) \cap C_o^{-}(\varepsilon).
$$

Here $C_i(\varepsilon)$, or $C_o(\varepsilon)$ is a double cone over B^{n-1}, or $(n-1)B^{n-1}$, respectively, with vertices $(0, \ldots, 0, \pm \tan \varepsilon)$. Moreover, $P(\varepsilon)$ is a convex polytope with m facets, all facet outer unit normals enclosing an angle ε with the $x_1 \ldots x_{n-1}$-hyperplane (actually with the respective facet outer unit normal of R^+ or R^-). If ε_0, and thus also ε, is sufficiently small, then still any $n-1$ facet outer unit normals of $P(\varepsilon)$ span an $(n-1)$-parallelotope of volume at least some $b' \in (0, b)$.

A routine calculation gives

$$
\frac{V(P(\varepsilon))}{S(P(\varepsilon))^{n/(n-1)}} \geq \frac{V(C_i(\varepsilon))}{S(C_o(\varepsilon))^{n/(n-1)}} = \frac{(\tan \varepsilon)^{1/(n-1)}}{n(2\kappa_{n-1})^{1/(n-1)}[1 + (n-1) \tan^2 \varepsilon]^{n/(2n-2)}}.
$$

Therefore,

$$
\begin{cases}
V(P(\varepsilon)) \geq S(P(\varepsilon))^{n/(n-1)} \cdot (\tan \varepsilon)^{1/(n-1)}/ \\
[n(2\kappa_{n-1})^{1/(n-1)}[1 + (n-1) \tan^2 \varepsilon_0]^{n/(2n-2)}] \geq \\
m^{-(n-2)/(n-1)} \left(\sum_{i=1}^m S_i(\varepsilon)^{n/(2n-2)} \right)^2 \cdot (\tan \varepsilon)^{1/(n-1)}/ \\
[n(2\kappa_{n-1})^{1/(n-1)}[1 + (n-1) \tan^2 \varepsilon_0]^{n/(2n-2)}].
\end{cases}
$$

Here $S_1(\varepsilon), \ldots, S_m(\varepsilon)$ are the areas of the facets of $P(\varepsilon)$, and the second inequality is equivalent to Hölder's inequality for the numbers $S_i(\varepsilon)$, between their arithmetic mean and their power mean with exponent $n/(2n-2) \in (0,1)$. Last observe $\tan \varepsilon_0 \in (0,1)$. $\qquad \square$

Now we can finish the third proof of Theorem 2, for $n > 3$. We proceed as for $n = 3$, but instead of Lemma 2 we use Lemma 4. In the last but one paragraph before Lemma 3, we have constructed a closed polygon M in the $(n-1)$-dimensional subspace X such that any $n-1$ normed side vectors $\overrightarrow{S_i}/S_i$ span a parallelotope of volume at least b. As in the third proof of Theorem 2, for $n = 3$, we take the first three consecutive vertices $A_1 A_2 A_3$ of M and "rotate" A_2 out of the subspace X, keeping the lengths of the two sides $A_1 A_2$ and $A_2 A_3$ fixed. We have a whole $(n-2)$-dimensional sphere on which A_2 can move, which intersects X orthogonally. By bounding the distance by which A_2 moves by a suitable threshold, we can ensure that any $n-1$ normed side vectors $\overrightarrow{S_i}/S_i$ still span a parallelotope of volume at least b', for some weaker bound $b' > 0$. The angle between $A_1 A_2$ or $A_2 A_3$ and the "horizontal" hyperplane X can be made as small as we like. Thus, Lemma 4 guarantees that the volume goes to zero as well. $\qquad\square$

5. Proofs for the hyperbolic case

For general concepts in hyperbolic geometry, we refer to [5, 7, 15, 32, 35, 38].

5.1. **Proof of Proposition 1.** Let H be the hyperplane of the facet of P of area S_m. Let $p\colon \mathbb{H}^n \to H$ be the orthogonal projection of $\mathbb{H}^n$ to H. We have that the image by p of the union of the $m-1$ facets, different from the above considered facet, contains the above considered facet.

Let dS be a surface element at a point $x \in \mathbb{H}^n$. Let its image by p be the surface element dS' at $p(x)$. Clearly it suffices to show that $dS' \leq dS$. We may suppose that dS is an (infinitesimal) $(n-1)$-ball, of radius dr, in the tangent space $T_x(\mathbb{H}^n)$ of $\mathbb{H}^n$ at x.

First we deal with the case when dS is orthogonal to the line $\ell(x, p(x))$ (for $x \in H$ we mean the line containing x and orthogonal to H). Then dS' is an infinitesimal $(n-1)$-ball in $T_{p(x)}(\mathbb{H}^n)$, of radius dr'. Let $h := |xp(x)|$. Then, by the trigonometric formulas of Lambert quadrangles in $\mathbb{H}^2$ (i.e., which have three right angles), we have (cf. [38, §29, (V)], or [14, Theorem 2.3.1]) $1 \leq \cosh h = \tanh(dr)/\tanh(dr')$. Hence $dr' \leq dr$, therefore $dS' \leq dS$.

Second we deal with the case when dS is not orthogonal to the line $\ell(x, p(x))$. Then the image by p of the infinitesimal $(n-1)$-ball dS in $T_x(\mathbb{H}^n)$ is an infinitesimal $(n-1)$-ellipsoid in $T_{p(x)}(\mathbb{H}^n)$, that has $n-2$ semiaxes equal to dr', and the $(n-1)$-st semiaxis smaller than dr'. Hence $dS' < dS$.

The case of equality is clear: the polytope must degenerate to the doubly counted facet of area S_m. $\qquad\square$

5.2. **Proof of Proposition 2.** From maximality of $S_m, S_{m-1}, \ldots, S_3$ there follows that all vertices lie at infinity — since a vertex cannot be incident only to the facets of areas S_2, S_1 — hence we have that also S_2, S_1 are maximal. $\qquad\square$

5.3. **Proof of Proposition 3.** Let P be a convex polyhedron as in the proposition, with respective facets $F_1, \ldots, F_m$. Let us consider any vertex of the facet F_i. We have for the angles of the facets incident to this vertex, that the angle of F_i at this vertex is at most the sum of the angles of all other facets incident to this vertex. In fact, intersecting P with an "infinitesimally small" sphere with centre at this vertex (in the conformal model), we obtain a convex spherical polygon with side lengths the

(convex) angles of all facets incident to this vertex, at this vertex (all these angles being in $[0, \pi)$).

Summing these inequalities we obtain the following. The sum t_1 of the angles of F_i is at most the sum t_2 of the angles of all other facets, having some common vertex with F_i, at the vertices of F_i. We increase t_2 if we take the sum t_3 of all angles of all other facets having some common vertex with F_i. We further increase t_3 if we take the sum t_4 of all angles of all facets different from F_i. Then the thus obtained inequality $t_1 \leq t_4$ is equivalent to the inequality to be proved.

Clearly, if we have at least one finite vertex, with incident edges not in a plane, then we have at least one strict inequality among the summed inequalities. So, in this case, we have strict inequality in the proposition. $\square$

The discussion of the inequality $t_1 \leq t_4$, from the proof of Proposition 3, for $\mathbb{R}^3$ and $\mathbb{S}^3$, cf. in Section 6.1, Remark 9.

5.4. **Proof of Theorem 3.**

Proposition 8 ([5, p. 127], [24, Theorem 1, Proposition 2]). *In $\mathbb{H}^n$, for $n \geq 2$, a simplex (with vertices at infinity admitted) is of maximal volume if and only if all its vertices are at infinity, and it is regular. It has a finite volume.*

Let v_n be the maximal volume of a simplex in $\mathbb{H}^n$. For instance, $v_2 = \pi$ and $v_3 = -3 \int\limits_{0}^{\pi/3} \log|2 \sin u| du = 1.0149416\ldots$, [35, p. 20], [5, p. 127]. Obviously, for the facet areas S_i of a compact simplex in $\mathbb{H}^n$ the inequalities

$$0 < S_1 \leq \ldots \leq S_{n+1} < v_{n-1}$$

hold.

Lemma 5. *Suppose that $\Delta ABC \subset \mathbb{H}^2$ is a triangle such that $\angle ACB = \pi/2$, and $|AC| = b$ and $|BC| = a$. Then the area S of this triangle satisfies the equality*

$$\tan S = \frac{\sinh a \cdot \sinh b}{\cosh a + \cosh b}.$$

This is a routine consequence of the trigonometric formulas for a right triangle in $\mathbb{H}^2$, using $S = \pi/2 - \alpha - \beta$, and $\tan(\angle CBA) = (\tanh b)/\sinh a$, and $\tan(\angle CAB) = (\tanh a)/\sinh b$, cf. [15, p. 238], and of $\tanh x = (\sinh x)/\cosh x$. $\square$

Lemma 6. *Let $d > 0$. Suppose that $\Delta ABC \subset \mathbb{H}^2$ is a triangle such that $|AB| \leq d$ and $|AC| \leq d$. Then the area S of this triangle satisfies the inequality*

$$S \leq 2 \arctan \frac{\cosh d - 1}{2\sqrt{\cosh d}}.$$

Proof. Without loss of generality we may suppose that $|AB| = |AC| = d$. Let H be the orthogonal projection of A to the line $\ell(B, C)$. Put $x = \cosh|AH|$, and $y = \cosh|BH| = \cosh|CH|$. Then $x \geq 1$ and $y \geq 1$, and $xy = \cosh d$. Let S be the area of ΔABC. By Lemma 5 we get

$$\tan^2(S/2) = \frac{(x^2 - 1)(y^2 - 1)}{(x + y)^2}.$$

Looking for the maximum of the numerator, and the minimum of the denumerator, we see that the maximal value of S is attained for $x = y = \sqrt{\cosh d}$, from which the lemma follows. $\qquad\square$

Lemma 7. *Let $F = (f_1, f_2)\colon P \to \mathbb{R}^2$ be a continuous function, where $P = [0, a] \times [0, b]$, with $a > 0$ and $b > 0$. Suppose that there are $u_1, u_2, v_1, v_2 \in \mathbb{R}$, such that $u_1 > u_2$ and $v_1 < v_2$, and*

$$f_1(x, 0) + f_2(x, 0) \leq 0, \quad f_1(x, b) + f_2(x, b) \geq 0, \quad and$$
$$u_1 f_1(0, y) + u_2 f_2(0, y) \leq 0, \quad v_1 f_1(a, y) + v_2 f_2(a, y) \leq 0,$$

for every $0 \leq x \leq a$ and $0 \leq y \leq b$. Then there is a point $(c, d) \in P$, such that $F(c, d) = (0, 0)$.

Proof. All is clear if F vanishes at some point of the boundary $\partial(P)$ of P. If F has no zero on $\partial(P)$, then it is sufficient to see that the index of the vector field F on the curve $\partial(P)$ is 1. Namely, this implies that there is (c, d) in the interior of P satisfying $F(c, d) = (0, 0)$ [25, p. 98, proof of Theorem VI.12, sufficiency].

To determine the index of F, we define the auxiliary function $F_0\colon \partial(P) \to \mathbb{S}^1$ as follows. We let $F_0\left([(a, 0), (a, b)]\right) = \{(1/\sqrt{2}, -1/\sqrt{2})\}$, and $F_0\left([(0, b), (0, 0)]\right) = \{(-1/\sqrt{2}, 1/\sqrt{2})\}$. Further, $F_0(x, b)$, or $F_0(x, 0)$, for x changing from b to 0, or from 0 to b, moves with constant angular velocity from $(1/\sqrt{2}, -1/\sqrt{2})$ to $(-1/\sqrt{2}, 1/\sqrt{2})$, or from $(-1/\sqrt{2}, 1/\sqrt{2})$ to $(1/\sqrt{2}, -1/\sqrt{2})$, in the half-plane $x + y \geq 0$, or $x + y \leq 0$, respectively. Then, for $(x, y) \in \partial(P)$, we have that $F(x, y)$ and $F_0(x, y)$ never point to opposite directions, hence $F(x, y)/\|F(x, y)\|, F_0(x, y)\colon \partial(P) \to \mathbb{S}^1$ are homotopic. Therefore the index of F equals the index of F_0, i.e., 1. $\qquad\square$

We still need two lemmas which together form a sharpening of two lemmas from [11]:

Lemma 8 ([11, Lemmas 1 and 2]). *Consider a (possibly degenerate) triangle A in $\mathbb{S}^2$, $\mathbb{R}^2$, or $\mathbb{H}^2$ with sides a, b, x, where $a, b > 0$. For the case of $\mathbb{S}^2$, we additionally assume $a + b \leq \pi$. Then, for a, b fixed, and $|a - b| \leq x \leq a + b$, the area $\bar{A}$ of this triangle is a concave function of x. (For $x = a + b = \pi$ on $\mathbb{S}^2$, we define $\bar{A}$ by a limit procedure, i.e., we set $\bar{A} = \pi$; observe that for $a + b = \pi$, the area $\overline{A}$ is half the area of a digon with sides containing the sides a, b of A.) In addition, the area is strictly concave for $\mathbb{R}^2$ and $\mathbb{H}^2$, and, for $a + b < \pi$, also for $\mathbb{S}^2$.* $\qquad\square$

We calculate more precise details about this concave function and the value of its maximum:

Lemma 9. *We use the notations of Lemma 8, and denote by γ the angle of the sides a, b. For $\mathbb{S}^2$ let us additionally suppose $a + b < \pi$. Then $\bar{A}$ equals 0 for $x = |a - b|$ and $x = a + b$, and it has a unique maximum for some value $x = x_{\max}$, with corresponding angle $\gamma = \gamma_{\max}$.*

For $\mathbb{H}^2$, we have

$$\cosh(x_{\max}/2) = \sqrt{(\cosh a + \cosh b)/2},$$
$$\cos \gamma_{\max} = \tanh(a/2) \cdot \tanh(b/2),$$

and the value of the maximum area is

$$\pi - 2 \arcsin\left(\sinh(a/2)/\sinh r\right) - 2 \arccos\left(\tanh(a/2)/\tanh r\right) +$$

$$\pi - 2\arcsin\left(\sinh(b/2)/\sinh r\right) - 2\arccos\left(\tanh(b/2)/\tanh r\right),$$

where $\cosh r = \sqrt{(\cosh a + \cosh b)/2}$.

For $\mathbb{R}^2$, *we have*

$$x_{\max}^2 = a^2 + b^2, \ \gamma_{\max} = \pi/2,$$

and the maximum area is $ab/2$.

For $\mathbb{S}^2$, *we have*

$$\cos(x_{\max}/2) = \sqrt{(\cos a + \cos b)/2},$$
$$\cos\gamma_{\max} = -\tan(a/2)\cdot\tan(b/2),$$

and the maximum area is

$$2\arcsin\left(\sin(a/2)/\sin r\right) + 2\arccos\left(\tan(a/2)/\tan r\right) - \pi +$$
$$2\arcsin\left(\sin(b/2)/\sin r\right) + 2\arccos\left(\tan(b/2)/\tan r\right) - \pi,$$

where $\cos r = \sqrt{(\cos a + \cos b)/2}$.

Moreover, letting $y/2$ *be the distance between the midpoint of the side* x *and the common vertex of the sides* a *and* b, *we have the following equivalences:*

$$\gamma \in [0, \gamma_{\max}) \Longleftrightarrow x < y, \ and \ \gamma = \gamma_{\max} \Longleftrightarrow x = y, \ and \ \gamma \in (\gamma_{\max}, \pi] \Longleftrightarrow x > y.$$

Proof. For $\mathbb{R}^2$ the statement is elementary. Therefore we investigate only the cases of $\mathbb{H}^2$ and $\mathbb{S}^2$.

Denote the vertices of the triangle opposite to the sides a, b, or x, by A, B, or C, respectively. Let D be the mirror image of C with respect to the midpoint of side x. Then the quadrangle $ABCD$ is centrally symmetric with respect to the intersection O of its diagonals BC (of length x) and AD (of length y). Its area is $2\bar{A}$, so it suffices to investigate its area.

We recall the isoperimetric property of the circle (in $\mathbb{R}^2, \mathbb{H}^2$, and on $\mathbb{S}^2$, but in the last case of radius $r < (a+b)/2 < \pi/2$) among sets of equal perimeter — namely that the maximum area is attained for the circle — for $\mathbb{S}^2$ for sets contained in some open half-$\mathbb{S}^2$, and considering the area of that domain having this boundary, that lies in the above open half-$\mathbb{S}^2$ [44, Ch. 18, §6, **2**, (18.39)]. Observe that a piecewise C^1 closed curve on $\mathbb{S}^2$, with length less than 2π, lies in some open half-$\mathbb{S}^2$, by elementary integral geometric considerations [44, Ch. 7, §2, (7.11), and Ch. 18, §6, **1**, (18.37)]. (For a very much detailed exposition of the isoperimetric inequality in spaces of constant curvature, i.e., $\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n$, cf. [45]; for some further details see [46].)

For $\gamma = 0$ we have $x = |a - b| < a + b = y$, while for $\gamma = \pi$ we have $x = a + b > |a - b| = y$. Therefore, for some $\gamma \in (0, \pi)$, we have $x = y$. This implies that, for this γ, i.e., for this x, we have that $ABCD$ is inscribed to a circle of centre O, and radius $r := x/2 = y/2$. This implies, via the isoperimetric property of the circle (on $\mathbb{S}^2$ of radius $r < (a+b)/2 < \pi/2$, in the above detailed sense) that this γ equals $\gamma_{\max}$, i.e., this x equals $x_{\max}$, [29, p. 63, Problem 21], [28, §5, Problem 63], [30, p. 52]. (These references deal with the case of $\mathbb{R}^2$, but their well-known proof takes over to $\mathbb{H}^2$ and $\mathbb{S}^2$, if we use the isoperimetric property of the circle, on $\mathbb{S}^2$ of radius $r < (a+b)/2 < \pi/2$, in the above detailed sense.)

We determine the radius of this circle. By the law of cosines for the triangles $\Delta AOC, \Delta DOC$, we have, writing $\varphi := \angle BOC$, for $\mathbb{H}^2$, that

$$\cosh a = \cosh(x/2) \cdot \cosh(y/2) - \sinh(x/2) \cdot \sinh(y/2) \cdot \cos \varphi,$$

and

$$\cosh b = \cosh(x/2) \cdot \cosh(y/2) + \sinh(x/2) \cdot \sinh(y/2) \cdot \cos \varphi.$$

Adding these, we obtain

$$\cosh a + \cosh b = 2 \cosh(x/2) \cdot \cosh(y/2). \tag{25}$$

Analogously, for $\mathbb{S}^2$, we obtain

$$\cos a + \cos b = 2 \cos(x/2) \cdot \cos(y/2).$$

(These are the analogues of the parallelogram law in $\mathbb{R}^2$.) Thus, for $\mathbb{H}^2$, we have

$$\cosh a + \cosh b = 2 \cosh^2 r = 2 \cosh^2(x_{\max}/2),$$

and, for $\mathbb{S}^2$, we have

$$\cos a + \cos b = 2 \cos^2 r = 2 \cos^2(x_{\max}/2),$$

while $0 < r \le (a+b)/2 < \pi/2$.

Further, for $\mathbb{H}^2$, we have that x is a strictly increasing function of γ, and, by (25), y is a strictly decreasing function of x. Hence, for $x = 2r$ we have $y = 2r$, for $|a - b| \le x < 2r$ we have $2r < y \le a + b$, and, similarly, for $2r < x \le a + b$ we have $|a - b| \le y < 2r$. These imply the last equivalences in the lemma for $\mathbb{H}^2$.

Next we determine $\cos \gamma_{\max}$ for $\mathbb{H}^2$. By the law of cosines, for the triangle ΔABC, we have

$$\cos \gamma_{\max} = \frac{\cosh a \cdot \cosh b - \cosh(2r)}{\sinh a \cdot \sinh b}.$$

Here $\cosh(2r) = 2 \cosh^2 r - 1 = \cosh a + \cosh b - 1$, and this implies the formula in the lemma. (Observe that $0 < a, b$ implies $0 < \tanh(a/2) \cdot \tanh(b/2) < 1$.)

For $\mathbb{S}^2$, the proof of the last equivalences in the lemma, and the calculation of $\cos \gamma_{\max}$ are analogous. (Observe that then $0 < a, b$ and $a/2 + b/2 < \pi/2$ imply $0 < \tan(a/2) \cdot \tan(b/2) < 1$.)

Last, the value of the maximum follows by the trigonometric formulas for a right triangle, in $\mathbb{H}^2$, and $\mathbb{S}^2$. $\qquad \square$

In order to prove Theorem 3 we need the following
Construction 1. Consider a number

$$S \in (0, \pi/2) \tag{26}$$

and a number $t > 0$, such that

$$2\sinh(t/2) > \tan S.$$

(Later S will be the area of a compact right triangle, which explains condition (26). At the same time, it explains the hypothesis $0 < S_4 < \pi/2$ of the theorem, since in the proof S will be chosen, e.g., as S_4.)

Now, we define a function

$$f_{t,S} \colon [0, t] \to \mathbb{R}$$

as follows. For any $x \in [0, t]$, consider the function

$$g_x(y) := \arctan \frac{\sinh x \cdot \sinh y}{\cosh x + \cosh y} + \arctan \frac{\sinh(t - x) \cdot \sinh y}{\cosh(t - x) + \cosh y},$$

where $y \in [0, \infty)$. It is easy to see that $(d/dy)g_x(y) > 0$ for $y \in [0, \infty)$, and $g_x(0) = 0$, and

$$\lim_{y \to \infty} g_x(y) = \arctan(\sinh x) + \arctan(\sinh(t - x))$$

$$\geq \arctan(\sinh x + \sinh(t - x)) \geq \arctan(2 \sinh(t/2)) > S$$

for all $x \in [0, t]$. Here, at the first inequality, we used concavity of the function $\arctan y$ on $[0, \infty)$ and $\arctan 0 = 0$, and, at the second inequality, we used convexity of the function $\sinh x$ on the interval $[0, t]$. Therefore, there is a unique $\tilde{y} \in (0, \infty)$ such that $g_x(\tilde{y}) = S$. We put

$$f_{t,S}(x) := \tilde{y} \in (0, \infty). \tag{27}$$

Now, we consider some properties of the defined function. Obviously, $f_{t,S}$ is continuous on $[0, t]$ (moreover, is C^1 on $(0, t)$), and $f_{t,S}(x) = f_{t,S}(t - x)$. It is easy to get a geometric interpretation of $f_{t,S}$.

Let us consider a triangle $\triangle ABC \subset \mathbb{H}^2$ with the following properties:
1) $|AB| = t$,
2) if H is the orthogonal projection of C to the line $\ell(A, B)$, then $|AH| = x \in [0, t]$,
3) the area of $\triangle ABC$ is S.
It is easy to see (using Lemma 5), that for such a triangle we get $|CH| = f_{t,S}(x)$.

It is easy to see that for $0 < \tilde{S} < S$ we have $f_{t,\tilde{S}}(x) < f_{t,S}(x)$ for every $x \in [0, t]$.

In what follows we determine the number

$$h_{t,S} := f_{t,S}(0) = f_{t,S}(t). \tag{28}$$

By Lemma 5, we have

$$\tan S = \frac{\sinh t \cdot \sinh h_{t,S}}{\cosh t + \cosh h_{t,S}}.$$

Solving this equation for $\cosh h_{t,S}$, we get

$$\cosh h_{t,S} = \frac{\tan^2 S \cdot \cosh t + \sqrt{1 + \tan^2 S} \cdot \sinh^2 t}{\sinh^2 t - \tan^2 S}.$$

(Observe that $\sinh t > 2 \sinh(t/2) > \tan S$ by strict convexity of the function $\sinh t$ on $[0, \infty)$ and $\sinh 0 = 0$.) From this we see that $\cosh h_{t,S} \to 1/\cos S$ as $t \to \infty$.

Proof of Theorem 3. **1.** First we consider the case when hypothesis (1) of Theorem 3 holds.

Let us take a $t > 0$, such that

$$2 \sinh(t/2) > \tan S_4, \tag{29}$$

and for the number $h_{t,S_4} := f_{t,S_4}(0) = f_{t,S_4}(t)$ (see (27) and (28)) we have

$$\frac{\cosh h_{t,S_4} - 1}{2\sqrt{\cosh h_{t,S_4}}} < \tan(S_1/2). \tag{30}$$

Such a t exists, since $\cosh h_{t,S_4} \to 1/\cos S_4$ as $t \to \infty$, and

$$\lim_{t\to\infty} \frac{\cosh h_{t,S_4} - 1}{2\sqrt{\cosh h_{t,S_4}}} = \frac{1 - \cos S_4}{2\sqrt{\cos S_4}} < \tan(S_1/2),$$

by hypothesis (1) of the theorem.

Consider any plane σ in $\mathbb{H}^3$. Take points $A_1, A_2 \in \sigma$ such that $|A_1 A_2| = t$, where t has been chosen above. Let σ^+ and σ^- be the two half-planes bounded by the line $\ell(A_1, A_2)$ in σ.

For a given $x \in [0, t]$, we take the point $H = H(x)$ on the segment $[A_1, A_2]$, such that $|A_1 H| = x$. Consider the half-line l_x from H in σ^+, that is orthogonal to the line $\ell(A_1, A_2)$. Now, let $\tilde{A}_4 = \tilde{A}_4(x)$ be the point on l_x satisfying $|\tilde{A}_4 H| = f_{t,S_3}(x)$, and let $\tilde{A}_3 = \tilde{A}_3(x)$ be the point on l_x satisfying $|\tilde{A}_3 H| = f_{t,S_4}(x)$. By $S_4 \geq S_3$, we have $\tilde{A}_4 \in [\tilde{A}_3, H]$, thus also $|H\tilde{A}_4| \leq |H\tilde{A}_3|$.

Now, let $\{\gamma(\varphi)\}$ be the one-parameter group of rotations about the line $\ell(A_1, A_2)$ through the angles φ (in some definite sense of rotation), with $\gamma(0)$ being the identity. For $\varphi \in [0, \pi]$, we consider the point $A_3 = A_3(x, \varphi) := \gamma(\varphi)(\tilde{A}_3(x))$. Note that $A_3(x, \pi) \in \sigma^-$. Consider also $A_4(x, \varphi) := \tilde{A}_4(x)$, and $A_1(x, \varphi) := A_1$, and $A_2(x, \varphi) := A_2$.

We are going to prove that there exists an $(x, \varphi) \in [0, t] \times [0, \pi]$ such that the (possibly degenerate) tetrahedron $T = T(x, \varphi) := A_1(x, \varphi) A_2(x, \varphi) A_3(x, \varphi) A_4(x, \varphi)$ has facet areas S_1, S_2, S_3, S_4 (for the facets opposite to $A_1(x, \varphi)$, etc.).

Let $s_i(x, \varphi)$ be the area of the facet of $T(x, \varphi)$ that is opposite to the vertex $A_i = A_i(x, \varphi)$. Obviously, $s_4(x, \varphi) = S_4$ and $s_3(x, \varphi) = S_3$, by our construction.

Let us define functions $f_1, f_2 : [0, t] \times [0, \pi] \to \mathbb{R}$ as follows:

$$f_1(x, \varphi) = s_2(x, \varphi) - S_2, \quad f_2(x, \varphi) = s_1(x, \varphi) - S_1. \tag{31}$$

It is easy to see that

$$f_1(x, 0) + f_2(x, 0) = S_4 - S_1 - S_2 - S_3 < 0, \tag{32}$$
$$f_1(x, \pi) + f_2(x, \pi) = S_3 + S_4 - S_1 - S_2 \geq 0. \tag{33}$$

Now we check that

$$f_1(0, \varphi) < 0, \quad f_2(t, \varphi) < 0, \tag{34}$$

for all $\varphi \in [0, \pi]$.

For the first inequality in (34), we note that the point $\tilde{A}_3(0)$ satisfies $|\tilde{A}_3(0)A_1| = f_{t,S_4}(0) = h_{t,S_4}$, and the point $\tilde{A}_4(0)$ satisfies $|\tilde{A}_4(0)A_1| = f_{t,S_3}(0) =: h_{t,S_3} \leq h_{t,S_4}$. Therefore, for every $\varphi \in [0, \pi]$, the triangle $\triangle A_1 A_3(0, \varphi) A_4(0, \varphi)$ satisfies $|A_1 A_3(0, \varphi)| \leq h_{t,S_4}$ and $|A_1 A_4(0, \varphi)| \leq h_{t,S_4}$. By Lemma 6 and (30), we get

$$s_2(0, \varphi) \leq 2 \arctan \frac{\cosh h_{t,S_4} - 1}{2\sqrt{\cosh h_{t,S_4}}} < S_1 \leq S_2.$$

Therefore, $f_1(0, \varphi) = s_2(0, \varphi) - S_2 \leq s_2(0, \varphi) - S_1 < 0$ for all $\varphi \in [0, \pi]$.

For the second inequality of (34), we replace in the above argument $A_1, \tilde{A}_3(0)$, or $\tilde{A}_4(0)$ by $A_2, \tilde{A}_3(t)$, or $\tilde{A}_4(t)$, respectively. Then we get $f_2(t, \varphi) = s_1(t, \varphi) - S_1 < 0$ for all $\varphi \in [0, \pi]$.

Taking into account the inequalities (32–34), by Lemma 7 (for $(u_1, u_2, v_1, v_2) := (1, 0, 0, 1)$), we get that there exists an $(x, \varphi) \in [0, t] \times [0, \pi]$, such that $f_1(x, \varphi) =$

$f_2(x, \varphi) = 0$. This means that $s_1(x, \varphi) = S_1$ and $s_2(x, \varphi) = S_2$, for the corresponding, possibly degenerate tetrahedron T.

2. Now we consider the case when hypothesis (2) of Theorem 3 holds. We use the same construction of the tetrahedron T, as in the first case. For the functions f_1 and f_2 defined by (31) we get the inequalities (32) and (33). Now we check that

$$f_2(0, \varphi) \geq 0, \quad f_1(t, \varphi) \geq 0, \tag{35}$$

for all $\varphi \in [0, \pi]$. For this we note that

$$s_1(0, \varphi) \geq s_1(0, 0) = S_4 - S_3 \geq S_2 \geq S_1, \tag{36}$$

and

$$s_2(t, \varphi) \geq s_2(t, 0) = S_4 - S_3 \geq S_2, \tag{37}$$

for $\varphi \in [0, \pi]$, provided t is sufficiently large, as we will prove. Of course, we have to prove only the first inequalities in (36) and (37).

We will investigate $s_1(0, \varphi)$; the case of $s_2(t, \varphi)$ is analogous. Recall that $|H\tilde{A}_4| \leq |H\tilde{A}_3|$, that implies $h_{0,S_3} = |A_1(0, \varphi)A_4(0, \varphi)| \leq |A_1(0, \varphi)A_3(0, \varphi)| = h_{0,S_4}$. For t fixed, but $\varphi \in [0, \pi]$ variable, we have for the third side of the triangle $\Delta A_1(0, \varphi)A_3(0, \varphi)A_4(0, \varphi)$ that

$$|A_3(0, \varphi)A_4(0, \varphi)| \in [h_{0,S_4} - h_{0,S_3}, h_{0,S_4} + h_{0,S_3}].$$

Therefore, to show

$$s_1(0, \varphi) \geq s_1(0, 0), \tag{38}$$

we must show that the area of the triangle with two sides $a := |A_2(0, \varphi)A_4(0, \varphi)|$ and $b := |A_2(0, \varphi)A_3(0, \varphi)|$, where $a \leq b$ (since $\cosh a = \cosh t \cdot \cosh |A_1(0, \varphi)A_4(0, \varphi)| \leq \cosh t \cdot \cosh |A_1(0, \varphi)A_3(0, \varphi)| = \cosh b$), and third side $c := |A_1(0, \varphi)A_3(0, \varphi)| - |A_1(0, \varphi)A_4(0, \varphi)| = h_{0,S_4} - h_{0,S_3}$, is less than or equal to the area of the triangle with the same first two sides, and with third side in the interval

$$[h_{0,S_4} - h_{0,S_3}, h_{0,S_4} + h_{0,S_3}] \subset [h_{0,S_4} - h_{0,S_3}, 2h_{0,S_4}] \subset [h_{0,S_4} - h_{0,S_3}, \text{const}].$$

For the last inclusion observe the following. By the geometric interpretation, for S_4 fixed, and $t > 2 \operatorname{arsinh}[(\tan S_4)/2]$ (cf. (29)) increasing, h_{0,S_4} decreases. Therefore, h_{0,S_4} remains bounded, for S_4 fixed, and t increasing from its originally chosen value, t_0, say, to infinity.

Inequality (38) will be proved if we show that, fixing the first two sides a and b, and varying the third side, x, say, in the interval $[h_{0,S_4} - h_{0,S_3}, \text{const}]$, the area is a monotonically increasing function of the length of the third side.

Now we apply Lemmas 8 and 9, for the triangle with sides a, b, x. We need to show that its area is increasing for $x \in [b - a, \text{const}]$, where, from the preceding considerations, we know that $0 \leq b - a \leq \text{const}$. By these Lemmas, this is satisfied for $x \in [b - a, x_{\max}]$ — which we can apply provided $b - a \leq \text{const} \leq x_{\max}$ — where $\cosh^2(x_{\max}/2) = (\cosh a + \cosh b)/2$. Thus, it suffices to show that $x_{\max} \geq \text{const}$, i.e., that $x_{\max} \to \infty$ for $t \to \infty$.

Now we estimate $x_{\max}$ from below. We have

$$\cosh^2(x_{\max}/2) = (\cosh a + \cosh b)/2 \geq \cosh a > e^a/2,$$

hence

$$(e^{x_{\max}/2})^2 > \cosh^2(x_{\max}/2) > e^a/2,$$

hence

$$x_{\max} > a - \log 2 \ge t - \log 2 \to \infty,$$

as we wanted to show. Thus, (35) is proved.

Taking into account the inequalities (32), (33), (35), by Lemma 7 (applied for $(u_1, u_2, v_1, v_2) = (0, -1, -1, 0)$), we get that there exists an $(x, \varphi) \in [0, t] \times [0, \pi]$, such that $f_1(x, \varphi) = f_2(x, \varphi) = 0$. This means that $s_1(x, \varphi) = S_1$ and $s_2(x, \varphi) = S_2$ for the corresponding, possibly degenerate tetrahedron T.

3. It remains to exclude degeneration of our tetrahedra. Our construction yields degenerate tetrahedra only for $\varphi = 0$ and $\varphi = \pi$. In the first case $S_4 = S_1 + S_2 + S_3$, a contradiction to our hypotheses. In the second case $S_4 + S_3 = S_2 + S_1$, that implies $\pi > S_4 = S_3 = S_2 = S_1 > 0$. (By the way, this can occur only in **1** of the proof of this theorem.) Then a suitable regular tetrahedron satisfies the conclusion of the theorem.   □

Remark 7. Let us apply the construction in the proof of Theorem 3 to the numbers $S_i \varepsilon^2$, and $t\varepsilon$, rather than S_i and t, where $\varepsilon \to 0$. Then, for sufficiently small $\varepsilon > 0$, (1) from Theorem 3 holds, and, as an analogue of (29), we have $2\sinh(t\varepsilon/2) > \tan(S_4\varepsilon^2)$. Then we obtain in the limit a Euclidean tetrahedron with facet areas S_i, and one edge t. This gives a new proof for the last statement of Theorem E for $\mathbb{R}^3$ (existence of tetrahedra of arbitrarily small positive volume). Namely, for $t \to \infty$, the heights of the facets meeting at the edge of length t, corresponding to this edge, are $O(1/t)$. Thus, the tetrahedron is included in a right circular cylinder, of height t, and radius of base $O(1/t)$. Hence, the volume of the tetrahedron is $O(1/t)$, while t is as large as we want. Degeneration is excluded as in Step **3** of the proof of Theorem 3.

6. Proofs for the spherical case

Recall our convention about the notion of *simplices in* $\mathbb{S}^n$, before Proposition 4.

6.1. **Proof of Proposition 4. 1.** We begin with the proof of the first inequality. Let the facets of P be F_i. We have for their areas

$$S_i = \text{const}_n \cdot \int |F_i \cap \mathbb{S}^1| d\mathbb{S}^1, \tag{39}$$

where the integration is taken with respect to the unique $O(n+1)$-invariant probability Borel measure (for the standard embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$) on the manifold of all the great-$\mathbb{S}^1$'s in $\mathbb{S}^n$ (cf. [44, Ch. 18, §6, 1]), where $|\cdot|$ denotes cardinality, and $\text{const}_n > 0$.

Observe that at the integration we may disregard those $\mathbb{S}^1$'s, which lie in the great-$\mathbb{S}^{n-1}$ spanned by the facet F_m, since they have measure 0. By the same reason, we may disregard those $\mathbb{S}^1$'s, which pass through the relative boundary of F_i (in the great-$\mathbb{S}^{n-1}$ spanned by it), for all $i \in \{1, \ldots, m\}$, simultaneously. If an $\mathbb{S}^1$ does not lie in the above hyperplane, and does not intersect the above relative boundaries, then it cannot contain two opposite points of any F_i (since F_i lies in a closed half-$\mathbb{S}^{n-1}$, both these points would lie in the relative boundary of F_i, taken with respect to the great-$\mathbb{S}^{n-1}$ spanned by it). If such an $\mathbb{S}^1$ enters P at some point $p \in F_m$, it must also leave P, through some other facet (since this $\mathbb{S}^1$ does not contain two opposite points of F_m), till it comes back to p. (This holds even in the degenerate

case, when some portion of F_m is a doubly counted boundary of P, either as a "flat" piece of P, or as bounded from both sides by the interior of P.)

This implies that, for $i = m$, the integral (39) is at most the sum of all analogous integrals for all $1 \le i \le m - 1$. Then (39) gives our inequality.

Now suppose that P lies in an open half-$\mathbb{S}^n$ (in the northern hemisphere, say), but does not degenerate to the doubly-counted F_m.

Let $\mathbb{S}_m$ be the great-$\mathbb{S}^{n-1}$ spanned by F_m. If $\cup_{i=1}^{n-1} F_i \not\subset \mathbb{S}_m$, then there exists an $x \in \cup_{i=1}^{m-1} \text{rel int } F_i$, such that $x \notin \mathbb{S}_m$. Also, there exists a $y \in \mathbb{S}_m \setminus F_m$ that also lies in the open northern hemisphere; then $x \ne y$. Then some neighbourhood of the great-$\mathbb{S}^1$ xy (since both x, y lie in the open northern hemisphere, this great-$\mathbb{S}^1$ exists), in the set of all great-$\mathbb{S}^1$'s in $\mathbb{S}^n$, has a positive measure. Thus the set of $\mathbb{S}^1$'s, intersecting $\cup_{i=1}^{m-1} F_i$, but not F_m, has a positive measure. This implies the strict inequality in this case.

If $\cup_{i=1}^{n-1} F_i \subset \mathbb{S}_m$, then, whether $\cup_{i=1}^{n-1} F_i \not\subset F_m$ or $\cup_{i=1}^{n-1} F_i \subset F_m$, we have strict inequality, unless P degenerates to the doubly counted facet F_m. This, however, was excluded.

2. We turn to the proof of the second inequality. We use the same formula (39). Now we disregard those $\mathbb{S}^1$'s that lie in the great-$\mathbb{S}^{n-1}$'s spanned by any facet of P, as well as those $\mathbb{S}^1$'s that pass through the relative boundary of any facet F_i. We compare the sum of the right hand sides of (39) for all $1 \le i \le m$, and the analogous integral, when in the right hand side of (39), we take a great-$\mathbb{S}^{n-1}$ rather than F_i.

Clearly, for a great-$\mathbb{S}^{n-1}$ the cardinality of its intersection with a great-$\mathbb{S}^1$ is almost always 2. For the $\mathbb{S}^1$'s that were not disregarded, and for any i, the cardinality $|F_i \cap \mathbb{S}^1|$ is at most 1, since a great-$\mathbb{S}^1$ cannot contain two opposite points of F_i (see part **1** of this proof). For P not degenerate, one great-$\mathbb{S}^1$ cannot intersect the interiors of three facets F_i. Namely, at each point of intersection it passes either outward, or inward P (with some definite orientation of our $\mathbb{S}^1$). Thus there would be at least four points of intersection, and the intersection of P and this great-$\mathbb{S}^1$ would be the union of at least two disjoint non-trivial arcs, contradicting convexity of P. Hence, the sum of the integrands in the right hand sides of (39), for $i = 1, \ldots, m$, is less than 3. For P degenerate, the same statement holds. This implies the second inequality of the proposition.

If P lies in an open half-$\mathbb{S}^n$, then the set of $\mathbb{S}^1$'s intersecting the boundary of the open half-$\mathbb{S}^n$, but not intersecting $\cup_{i=1}^m F_i$, has a positive measure. (Namely, any great-$\mathbb{S}^1$, sufficiently close to the boundary of the open half-$\mathbb{S}^n$, has this property.) This implies the strict inequality in this case. $\square$

Remark 8. For Proposition 1, for $\mathbb{H}^n$, the same proof works as in **1** of the proof of Proposition 4, however, without the case of equality. We have to consider also [44, Ch. 18, §6, 1], taking, rather than $\mathbb{S}^1$, a segment of a fixed positive length t, and then letting t tend to infinity. However, we preferred to give the elementary proof for Proposition 1.

Remark 9. Clearly, the inequality $t_1 \le t_4$ from the proof of Proposition 3 is valid also for $\mathbb{R}^3$ and $\mathbb{S}^3$. However, for $\mathbb{R}^3$, and also for $\mathbb{S}^3$, provided that each facet is contained in a closed half-$\mathbb{S}^2$, and has at least three sides — in particular, if the polyhedron is contained in an open half-$\mathbb{S}^3$ — they are evident. For $\mathbb{R}^3$, the sum of the angles of F_i is $t_1 = (k_i - 2)\pi$, while it has k_i neighbouring faces, each having sum of its angles at least π, so $t_1 = (k_i - 2)\pi < k_i\pi \le t_4$. Similarly, for $\mathbb{S}^3$, with

the above hypotheses, the sum of the angles of F_i is $t_1 = S_i + (k_i - 2)\pi$, while each other facet F_j has a sum of its angles $S_j + (k_j - 2)\pi \geq S_j + \pi$. So the sum of the angles of these facets F_j is at least $\sum_j (S_j + \pi)$, hence $(\sum_j S_j) + k_i\pi \leq t_4$. Therefore, $t_1 = S_i + (k_i - 2)\pi < S_i + k_i\pi \leq (\sum_j S_j) + k_i\pi \leq t_4$. Here we used Proposition 4, first inequality, that implies $S_i \leq \sum_j S_j$, provided all facets lie in some closed half-$\mathbb{S}^2$'s.

6.2. **Proof of Theorem 4.** Now we give the spherical analogues of Lemmas 5 and 6.

Lemma 10. *Suppose that $\triangle ABC \subset \mathbb{S}^2$ is a triangle such that $\angle ACB = \pi/2$, $|AC| = b$ and $|BC| = a$, where $0 < a, b \leq \pi$. Then the area S of this triangle satisfies the equality*

$$\tan S = \frac{\sin a \cdot \sin b}{\cos a + \cos b}$$

if $a + b \neq \pi$. For $a + b = \pi$, the area is $S = \pi/2$.

Lemma 11. *Let $0 < d \leq \pi/2$. Suppose that $\triangle ABC \subset \mathbb{S}^2$ is a triangle such that $|AB| \leq d$ and $|AC| \leq d$. Then the area S of this triangle satisfies the inequality*

$$S \leq 2\arctan\frac{1 - \cos d}{2\sqrt{\cos d}}$$

if $d \neq \pi/2$. For $d = \pi/2$, the bound is $S \leq \pi$.

Proof of Lemmas 10 and 11. We proceed analogously as in Lemmas 5 and 6. (Observe that for $a + b = \pi$ the statement of Lemma 10 is elementary, so we may suppose $a + b \neq \pi$. Similarly, for $d = \pi/2$ the statement of Lemma 11 is elementary, so we may suppose $d < \pi/2$, thus $\cos d > 0$.) $\qquad\square$

To prove Theorem 4, we will need an analogous construction as for Theorem 3.
Construction 2. Let

$$S \in (0, \pi/2], \tag{40}$$

and let us choose

$$t = \pi/2.$$

(Later we will apply Lemma 8, with sides a, b at most $\pi/2$, and Lemma 8 is false for $a = b \in (t, \pi) = (\pi/2, \pi)$. Thus $t = \pi/2$ is the maximal value, for which our proof applies. Later, S will be the area of a spherical triangle included in a spherical triangle with three sides $\pi/2$, which explains (40).) Now, we define a function

$$f_{t,S} \colon [0, t] \to \mathbb{R}$$

as follows. For any $x \in [0, t]$, consider the function

$$g_x(y) := \arctan\frac{\sin x \cdot \sin y}{\cos x + \cos y} + \arctan\frac{\sin(t - x) \cdot \sin y}{\cos(t - x) + \cos y},$$

defined for $y \in [0, \pi/2]$. It is easy to see that $(d/dy)g_x(y) > 0$ for $y \in [0, \pi/2)$, $g_x(0) = 0$, and

$$g_x(\pi/2) = \pi/2 \geq S,$$

for all $x \in [0, t]$. Therefore, there is a unique $\tilde{y} \in (0, \pi/2]$ such that $g_x(\tilde{y}) = S$. We put

$$f_{t,S}(x) := \tilde{y} \in (0, \pi/2].$$

Now, we consider some properties of the defined function. Obviously, $f_{t,S}$ is continuous on $[0,t]$ (moreover, is C^1 on $(0,t)$), and $f_{t,S}(x) = f_{t,S}(t-x)$. It is easy to get a geometric interpretation of $f_{t,S}$.

Let us consider a triangle $\Delta ABC \subset \mathbb{S}^2$ with the following properties:

1) $|AB| = t$,

2) C has an orthogonal projection H to the line $\ell(A, B)$, lying in the segment $[A, B]$, such that $|AH| = x \in [0, t]$ (observe that there are at least two orthogonal projections of C to $\ell(A, B)$),

3) the area of ΔABC is S.

It is easy to see (using Lemma 10), that for such a triangle we get $|CH| = f_{t,S}(x)$.

It is easy to see that for $0 < \tilde{S} < S$ we have $f_{t,\tilde{S}}(x) < f_{t,S}(x)$ for every $x \in [0, t]$.

In what follows we determine the number

$$h_{t,S} := f_{t,S}(0) = f_{t,S}(t).$$

From the geometric interpretation, it is the third side of a spherical triangle with two other sides of length $\pi/2$, and area S, i.e.,

$$h_{t,S} = S.$$

Proof of Theorem 4. **1.** First we consider the case when hypothesis (4) of Theorem 4 holds. We roughly follow the lines of the proof of Theorem 3, case when hypothesis (1) holds.

We have

$$\frac{1 - \cos h_{t,S_4}}{2\sqrt{\cos h_{t,S_4}}} = \frac{1 - \cos S_4}{2\sqrt{\cos S_4}} \leq \tan(S_1/2), \tag{41}$$

by the hypothesis of the theorem. From this point onwards the example is identical as in Theorem 3.

Still we have to show that our tetrahedron satisfies our convention before Proposition 4 about what do we mean by simplices in $\mathbb{S}^n$. There it is written that if we have a convex combinatorial simplex in an open half-$\mathbb{S}^n$, we consider it as a simplex in $\mathbb{S}^n$. Let $A_1 := \mathbf{e_1}$ and $A_2 := \mathbf{e_2}$ ($\mathbf{e_i}$ are the usual basic unit vectors). Then, rotation about $\ell(A_1, A_2)$ means replacement of $A_3 := \mathbf{e_3}$ by, say, $\mathbf{e_3}\cos\varphi + \mathbf{e_4}\sin\varphi$. Then T is in the closed half-$\mathbb{S}^3$ defined by the inequality $x_1 + x_2 \geq 0$ (the x_i's are the usual coordinates in $\mathbb{R}^4$). Even, for $S_3 \leq S_4 < \pi/2$, we have that T is contained in the open half-$\mathbb{S}^3$ defined by the inequality $x_1 + x_2 > 0$, and then we are done. For $S_3 < S_4 = \pi/2$, a slight perturbation of the above open half-$\mathbb{S}^3$ contains T, for all $\varphi \in [0, \pi]$. For $S_3 = S_4 = \pi/2$, and $0 \leq \varphi < \pi/2$ given, also a slight perturbation of the open half-$\mathbb{S}^3$ given by $x_1 + x_2 > 0$ contains T, and we are done. For $S_3 = S_4 = \varphi = \pi/2$, see part **3** of this proof.

We have to observe that, from the construction, we have $|A_3(x, \varphi)A_4(x, \varphi)| \leq f_{t,S_3}(x) + f_{t,S_4}(x) \leq 2f_{t,S_4}(x) \leq \pi$. Thus the edge $[A_3(x, \varphi), A_4(x, \varphi)]$ of our tetrahedron is in the closed angular domain swept by $\gamma(\varphi)\sigma^+$, for $\varphi \in [0, \pi]$, as it was in the hyperbolic case. (This explains the inequality $S_4 \leq \pi/2$ of the theorem — and thus also the inequality $S \leq \pi/2$ in the construction: without this inequality of the theorem, the last sentence would not be valid. Moreover, for $S_3 = S_4 \in (\pi/2, \pi)$, and for φ sufficiently close to π, we would have that $|A_3(x, \varphi)A_4(x, \varphi)|$, when defined not as a distance, but by analytic continuation from φ's close to 0, i.e., by retaining the geometry of the figure, is greater than π, which would imply that T, also defined

by analytic continuation, i.e., by retaining the geometry of the figure, would not be convex.)

We define $s_i(x, \varphi)$ (for $1 \leq i \leq 4$), and $f_i(x, \varphi)$ (for $i = 1, 2$) as in the proof of Theorem 3. The formulas

$$f_1(x, 0) + f_2(x, 0) = S_4 - S_1 - S_2 - S_3 < 0, \tag{42}$$

$$f_1(x, \pi) + f_2(x, \pi) = S_3 + S_4 - S_1 - S_2 \geq 0 \tag{43}$$

follow as (32) and (33) in the hyperbolic case. The formulas

$$f_1(0, \varphi) \leq 0, \quad f_2(t, \varphi) \leq 0,$$

for all $\varphi \in [0, \pi]$, follow from Lemma 11 (similarly, like in the hyperbolic case from Lemma 6; observe that here we have non-strict inequalities, since in (4) from Theorem 4 and (41) we have non-strict inequalities, while for the hyperbolic case in (1) from Theorem 3 and (30) we had strict inequalities). Then, like in the hyperbolic case, we choose $(u_1, u_2, v_1, v_2) := (1, 0, 0, 1)$, and finish the proof of case **1** like in the hyperbolic case, with the tetrahedron T being possibly degenerate. (Observe that allowing $S_4 > \pi/2$ we could have $h_{t,S_4} > \pi/2$; then $|A_1 A_3(0, \varphi)|, |A_1 A_4(0, \varphi)| \leq h_{t,S_4}$ does not render it possible to apply Lemma 11. This explains once more the inequality $S_4 \leq \pi/2$ of the theorem — and thus also the inequality $S \leq \pi/2$ in the construction — for case **1**.)

2. Now we consider the case when hypothesis (5) of Theorem 4 holds. We roughly follow the lines of the proof of Theorem 3, when hypothesis (2) holds. Like in Step **1** of this proof, we have that our tetrahedron satisfies our convention about what we mean by a simplex in $\mathbb{S}^n$, before Proposition 4, unless $S_3 = S_4 = \varphi = \pi/2$; in this case refer to Step **3** of this proof. We get the inequalities (42) and (43) as (32) and (33) in the hyperbolic case.

Now we check that

$$f_2(0, \varphi) \geq 0, \quad f_1(t, \varphi) \geq 0,$$

for all $\varphi \in [0, \pi]$. Like in the hyperbolic case, this reduces to showing that

$$s_1(0, \varphi) \geq s_1(0, 0), \quad s_2(t, \varphi) \geq s_2(t, 0). \tag{44}$$

We will investigate $s_1(0, \varphi)$; the case of $s_2(t, \varphi)$ is analogous. Observe that $|A_3(x, \varphi) A_4(x, \varphi)|$ is a strictly increasing function of $\varphi \in [0, \pi]$, with

$$|A_3(x, 0) A_4(x, 0)| = h_{t,S_4} - h_{t,S_3} = S_4 - S_3,$$

and

$$|A_3(x, \pi) A_4(x, \pi)| = h_{t,S_4} + h_{t,S_3} = S_4 + S_3.$$

By the geometric interpretation, we have

$$s_1(0, \varphi) = \angle A_3(0, \varphi) A_2(0, \varphi) A_4(0, \varphi) = |A_3(0, \varphi) A_4(0, \varphi)|,$$

that is strictly increasing for $\varphi \in [0, \pi]$. This shows (44). Then, like in the hyperbolic case, we choose $(u_1, u_2, v_1, v_2) := (0, -1, -1, 0)$, and finish the proof of **2** like in the hyperbolic case, with the tetrahedron T being possibly degenerate.

3. It remains to exclude degeneration of our tetrahedra. This happens exactly as in the hyperbolic case in Theorem 3 (here even we can have $\pi \geq S_4 = S_3 = S_2 = S_1 > 0$). We only observe that for $S_i = \pi/2$ (for each $1 \leq i \leq 4$) the regular tetrahedron is the intersection of $\mathbb{S}^3$ and an orthant, and then $\varphi = \pi/2$ (cf. the end of the third paragraph of **1**). □

6.3. **Proof of Proposition 5. 1.** For $n = 2$ one has a convex m-gon in a closed half-$\mathbb{S}^2$, of sides $S_1, \ldots, S_m$, and in fact, in an open half-$\mathbb{S}^2$, with strictly convex angles, and non-degenerate, if $S_m < S_1 + \ldots + S_{m-1}$, and $S_1 + \ldots + S_m < 2\pi$. (For the degenerate cases, i.e., when $S_m = S_1 + \ldots + S_{m-1}$, or $S_1 + \ldots + S_m = 2\pi$, we have a doubly counted segment, or a great-$\mathbb{S}^1$, respectively. If both equalities hold, we have also a digon. If both inequalities are strict, then we can copy the well-known proof in [30, pp. 53–54] — given there for the case of $\mathbb{R}^2$ — to obtain the existence of such a convex m-gon: one gets a polygon inscribed in a circle.)

For $\mathbb{S}^3$ we consider its equator $\mathbb{S}^2$, and each side of the above polygon is replaced by a facet, which is the union of all meridians (length equal to π) meeting that side. The vertices are replaced similarly by edges, which are meridians meeting these vertices. Additionally there are two new vertices at the North and South Poles. Then the ratio of the areas of the spherical digons and the lengths of the corresponding edges of our polygon is $V_2(\mathbb{S}^2)/V_1(\mathbb{S}^1)$, where V_i denotes i-volume. Moreover, the dihedral angles are the same as for the spherical m-gon in $\mathbb{S}^2$.

The inductive step is performed analogously for all $n > 3$ as well. The other stated properties are obvious.

2. For $n = 2$ we have a spherical triangle. We proceed by induction. Let $n \geq 3$, and let us assume that the statement of the theorem holds for $n - 1$. Then the numbers $(S_1 + S_2)V_{n-2}(\mathbb{S}^{n-2})/V_{n-1}(\mathbb{S}^{n-1})$, $S_3 V_{n-2}(\mathbb{S}^{n-2})/V_{n-1}(\mathbb{S}^{n-1})$, ..., $S_{n+1}V_{n-2}(\mathbb{S}^{n-2})/V_{n-1}(\mathbb{S}^{n-1})$ satisfy the hypotheses of the proposition for $n - 1$ (cf. the second proof of Theorem 2, with the modification in the second case there, that here $n + 1 \geq 4$, and thus $T_{\text{new}} := T_1 + T_2 \leq T_3 + \ldots + T_{n+1}$).

Therefore, we have on $\mathbb{S}^{n-1}$ a polyhedral complex, which is a combinatorial simplex, with these facet areas. Again, considering $\mathbb{S}^{n-1}$ as the equator of $\mathbb{S}^n$, all facets, and also all lower dimensional faces of the polyhedral complex on $\mathbb{S}^{n-1}$ are replaced by the union of all meridians (length equal to π) meeting that facet, or lower dimensional face. Thus we obtain facets, and also lower dimensional faces, of one dimension higher than the original ones. Additionally, there are two new vertices at the North and South Poles. Thus we obtained a polyhedral complex on $\mathbb{S}^n$, with facet areas $S_1 + S_2, S_3, \ldots, S_{n+1}$.

This last polyhedral complex has a facet, of area $S_1 + S_2$. The polyhedral complex has two vertices at the two poles, and n edges joining these two vertices. The facet of area $S_1 + S_2$ has $n - 1$ edges. On each of these $n - 1$ edges we add an extra vertex, at the same geographic latitude, and add an extra simplicial $(n - 2)$-face with these vertices, lying in the great-$\mathbb{S}^{n-2}$ spanned by these vertices, as well as all its faces of all lower dimensions. For a suitable choice of the latitude, the facet of area $S_1 + S_2$ is subdivided to two $(n - 1)$-dimensional simplicial facets of areas S_1 and S_2. These two facets, with their all lower dimensional faces are added as well. Each other facet (of areas $S_3, \ldots, S_{n+1}$) has thus one $(n - 2)$-face subdivided into two simplicial $(n - 2)$-faces — which, with all their lower dimensional faces have already been added above — thus these other facets also become combinatorial $(n - 1)$-simplices (by induction with respect to n).

The other stated properties follow by the construction. □

References

[1] A. D. Aleksandrov, *Zur Theorie der gemischten Volumina von konvexen Körpern III, Die Erweiterung zweier Lehrsätze Minkowski's über die konvexen Polyeder auf die beliebigen konvexen Körper*, Mat. Sb. **3** (1938), 27–46, Russian, with German summary, Zbl.**18.**42402.

[2] A. D. Alexandrow, *Die innere Geometrie der konvexen Flächen*, Math. Lehrbücher und Monographien, II. Abteilung: Math. Monographien, **4**, Akademie Verlag, Berlin, 1955, MR**17,**74.

[3] A. D. Alexandrow, *Konvexe Polyeder*, Math. Lehrbücher und Monographien, II. Abteilung: Math. Monographien, **8**, Akademie Verlag, Berlin, 1958, MR**19,**1192.

[4] V. A. Aleksandrov, *How to crumple a regular tetrahedral packet of milk, so that it could contain more* (Russian), Sorosovskij Obrazovatel'nyj Zhurnal **6** (2000), 121–127.

[5] D. V. Alekseevskij, E. B. Vinberg, A. S. Solodovnikov, *Geometry of spaces of constant curvature,* In: Geometry II (Ed. E. B. Vinberg), Enc. Math. Sci. Vol. **29**, 1–138, Springer, Berlin, 1993, MR**95b:**53042.

[6] F. C. Auluck, *The volume of a tetrahedron, the areas of the faces being given*, Proc. Indian Acad. Sci., Sect. A **7** (1938), 279–281, Zbl.**18.**37106, also available at http://www.ias.ac.in/jarch/proca/**7**, 300–302, or http://www.ias.ac.in/j_archive/proca/7/4/279-281/viewpage.html.

[7] R. Baldus, *Nichteuklidische Geometrie, Hyperbolische Geometrie der Ebene*, 4th ed., Bearb. und ergänzt von F. Löbell, Sammlung Göschen **970/970a**, de Gruyter, Berlin, 1964, MR**29#**3936.

[8] D. D. Bleecker, *Volume increasing isometric deformations of convex polyhedra*, J. Diff. Geom. **43** (1996), 505–526, MR**97g:**52035.

[9] C. W. Borchardt, *Über die Aufgabe des Maximums, welche der Bestimmung des Tetraeders von grössstem Volumen bei gegebenen Flächeninhalten der Seitenflächen für mehr als drei Dimensionen entspricht*, Math. Abh. Akad. Wiss. Berlin, 1866, 121–155 = C. W. Borchardts Gesämmelte Werke, Verl. G. Leimer, Berlin, 1888, 201–232. JFM**20.**15.01

[10] K. Böröczky, I. Bárány, E. Makai, Jr., J. Pach, *Maximal volume enclosed by plates and proof of the chessboard conjecture*, Discr. Math. **60** (1986), 101–120, MR**87m:**52016.

[11] K. Böröczky, G. Kertész, E. Makai, Jr., *The minimum area of a simple polygon with given side lengths*, Periodica Mathematica Hungarica **39** (1–3) (1999), 33–49, MR**2001e:**51016.

[12] P. Brass, W. Moser, J. Pach, *Research problems in discrete geometry*, Springer, New York, NY, 2005, MR**2006i:**52001.

[13] Ju. D. Burago, V. A. Zalgaller, *Isometric piecewise-linear embeddings of two-dimensional manifolds with a polyhedral metric into* $\mathbb{R}^3$, Algebra i Analiz **7** (1995) (3), 76–95 (in Russian); English translation in St. Petersburg Math. J. **7** (1996) (3), 369–385, MR**96g:**53091.

[14] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics **106**, Birkhäuser, Boston, Mass., 1992, MR**93g:**58149.

[15] H. S. M. Coxeter, *Non-Euclidean geometry. 6th ed.*, Spectrum Series, The Mathematical Association of America, Washington, DC, 1998, MR**99c:**51002.

[16] N. Dunford, J. T. Schwartz, *Linear operators, Vol. 1, General theory*, Pure and Appl. Math. **7**, Interscience, New York-London, 1958, MR**22#**8302.

[17] A. L. Edmonds, M. Hajja, H. Martini, *Orthocentric simplices and their centers*, Results Math. **47** (2005), 266–295, MR**2006e:**51021.

[18] L. Fejes Tóth, *The isepiphan problem for n-hedra*, Amer. J. Math. **70** (1978), 174–180, MR**9,**460.

[19] L. Fejes Tóth, *Reguläre Figuren*, Akadémiai Kiadó, Budapest, 1965, MR**30#**3408.

[20] W. Fenchel, B. Jessen, *Mengenfunktionen und konvexe Körper*, Danske Vid. Selsk., Mat.-Fys. Medd. **16** (1938), 1–31, Zbl.**18.**42401.

[21] L. Gerber, *The orthocentric simplex as an extreme simplex*, Pacific J. Math. **56** (1975), 97–111, MR**51**#12717.

[22] M. Goldberg, *The isoperimetric problem for polyhedra*, Tôhoku Math. J. **40** (1935), 226–236, Zbl.**10.**41004.

[23] P. Gritzmann, J. M. Wills, D. Wrase, *A new isoperimetric inequality*, J. reine angew. Math. **379** (1987), 22–30, MR**88h:**52018.

[24] U. Haagerup, H. J. Munkholm, *Simplices of maximal volume in hyperbolic n-space*, Acta Math. **147** (1981), 1–11, MR**82j:**53116.

[25] W. Hurewicz, H. Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton, N.J., 1941, MR**3,**312b.

[26] K. S. K. Iyengar, *A note on Narasinga Rao's problem relating to tetrahedra*, Proc. Indian Acad. Sci., Sect. A **7** (1938), 269–278, Zbl.**18.**37105, also available at http://www.ias.ac.in/jarch/proca/**7**, 290–299, or http://www.ias.ac.in/j_archive/proca/7/4/269-278/viewpage.html.

[27] K. S. K. Iyengar, K. V. Iyengar, *On a problem relating to a tetrahedron*, Proc. Indian Acad. Sci., Sect. A **7** (1938), 305–311, Zbl.**19.**07506, also available at http://www.ias.ac.in/jarch/proca/**7**, 326–332, or http://www.ias.ac.in/j_archive/proca/7/5/305-311/viewpage.html.

[28] I. M. Jaglom, W. G. Boltjanski, *Konvexe Figuren*, Hochschulbücher für Mathematik **24**, VEB Deutscher Verl. Wiss. Berlin, 1956, MR**18**,146f (MR**14**,197d).

[29] N. D. Kazarinoff, *Geometric inequalities*, Random House, New York-Toronto, 1961, New Mathematical Library **4**, MR**24**#**A**1.

[30] D. A. Kryžanovskij, *Isoperimetry: maximum and minimum properties of geometrical figures*, 3-rd ed., with the redaction of I. M. Jaglom (in Russian), Fizmatgiz, Moskva, 1959.

[31] J. L. Lagrange, *Solutions analytiques de quelques problèmes sur les pyramides triangulaires*, Nouv. Mém. Acad. Sci. Berlin, 1773, 149–176 = J. L. Lagrange, Œuvres complètes de Lagrange, Tome 3, Publié par J.-A. Serret, Gauthier-Villars, Paris, 1869, 659–692.

[32] H. Liebmann, *Nichteuklidische Geometrie*, 3rd ed., de Gruyter, Berlin, 1923, JFM**49.**0390.01.

[33] A. D. Milka, *Linear bendings of regular convex polyhedra* (Russian, with English summary), Mat. Fiz. Anal. Geom. **1** (1994), 116–130, MR**98k:**52056.

[34] A. D. Milka, V. A. Gor'kavyj, *Volume increasing bendings of regular polyhedra* (Russian, with English summary), Zb. Pr. Inst. Mat. NAN Ukr. **6** (2009), 152–182, Zbl.**1199,**52007.

[35] J. Milnor, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc., New Ser. **6** (1982), 9–24, MR**82m:**57005.

[36] Yu. V. Nikonorova, *On an isoperimetric problem on the Euclidean plane*, Trudy Rubtsovsk. Ind. Inst. **9** (2001), 66–72 (Russian), Zbl.**1006.**51014.

[37] I. Pak, *Inflating the cube without stretching*, Amer. Math. Monthly **115** (2008), 443–445,

[38] O. Perron, *Nichteuklidische Elementargeometrie der Ebene*, Math. Leitfäden, Teubner, Stuttgart, 1962, MR**25**#2489.

[39] A. V. Pogorelov, *Unique determination of convex surfaces*, Trudy Mat. Inst. i. V. A. Steklova, **29**, Izdat. Akad. Nauk SSSR, Moskva-Leningrad, 1949, Zbl. **41.**508.

[40] A. V. Pogorelov, *Geometric theory of stability of shells (Russian),* Modern Problems of Math., Nauka, Moskva, 1966, MR**34**#3843.

[41] A. V. Pogorelov, *Geometrical methods in the non-linear theory of elastic shells*, Nauka, Moskva, 1967, MR**36**#3535.

[42] A. Narasinga Rao, The Mathematics Student, June 1937, **5** (2), 90.

[43] I. S. Sabitov, *Algebraic methods for the solution of polyhedra*, Uspekhi Mat. Nauk **66** (2011) (3) (399), 3–66 (Russian); Transl. in: Russian Math. Surveys, **66** (3) (2011), 445–505, MR**2012k:**52027.

[44] L. A. Santaló, *Integral Geometry and Geometric Probability*, Encyclopedia of Mathematics and its Applications, **1**, Addison-Wesley, Reading, Mass. etc., 1976, MR**55**#6340.

[45] E. Schmidt, *Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie I*, Math. Nachr. **1** (1948), 81–157, MR**10,**471d.

[46] E. Schmidt, *Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie II*, Math. Nachr. **2** (1949), 171–244, MR**11,**534l.

[47] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, **44**, Cambridge University Press, Cambridge, 1993, MR**94d:**52007.

[48] E. Steinitz, *Polyeder und Raumeinteilungen,* Enzyklopädie der Math. Wiss., mit Einschluss ihrer Anwendungen, 3-er Band, Geometrie, Teil I, 2, Teubner, Leipzig, 1914-1931, Ch. 12, 1–139.

[49] T. Tarnai, Z. Gáspár, A. Lengyel, paper under preparation.

[50] T. Tarnai, K. Hincz, A. Lengyel, *Volume increasing inextensional deformation of a cube*, Proc. Internat. Assoc. for Shell and Spatial Structures (IASS) Symp. 2013, "Beyond the limits of man", 23-27 Sept., Wrocław Univ. of Techn., Poland.

[51] T. Tarnai, K. Hincz, A. Lengyel, paper under preparation.

[52] A. Tarski, *A decision method for elementary algebra and geometry*, 2-nd ed., Univ. Calif. Press, Berkeley, Los Angeles, Calif., 1951, MR**13,**423 (MR**10,**499).

[53] K. Venkatachaliengar, *On a problem of the tetrahedron*, Proc. Indian Acad. Sci., Sect. A **7** (1938), 257–260, Zbl.**18.**37107, also available at http://www.ias.ac.in/jarch/proca/**7**, 278–281, or http://www.ias.ac.in/j_archive/proca/7/4/257-260/viewpage.html.

N. V. Abrosimov, Sobolev Institute of Mathematics, Novosibirsk, Acad. Koptyug Av. 4, 630090, RUSSIA
  E-mail address: abrosimov@math.nsc.ru

E. Makai, Jr., A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, H-1364 Budapest, Pf. 127, HUNGARY
  E-mail address: makai@renyi.mta.hu, http://www.renyi.mta.hu/~makai

A. D. Mednykh, Sobolev Institute of Mathematics, Novosibirsk, Acad. Koptyug Av. 4, 630090, RUSSIA
  E-mail address: mednykh@math.nsc.ru

Yu. G. Nikonorov, South Mathematical Institute of the V. S. C. of the Russian Academy of Sciences, Vladikavkaz, Markus Str. 22, 362027, RUSSIA
  E-mail address: nikonorov2006@mail.ru

G. Rote, Institut für Informatik, Freie Universität Berlin, Takustr. 9, 14159 Berlin, GERMANY
  E-mail address: rote@inf.fu-berlin.de